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Kamāl al-Dīn al-Fārsī's additions to Book XIII of al-Ṭūsī's *Taḥrīr*

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Abstract

This article is devoted to Kamāl al-Dīn al-Fārsī's (d. 1319) additions to some remarks contained in Book XIII of al-Ṭūsī's *Taḥrīr uṣūl al-handasa*, concerning the construction of a semi-regular polyhedron inscribed into a sphere using the movement as a way for the construction. This treatise is one treatise among ten found in a codex preserved at the Bibliothèque nationale de Tunis.

Keywords: Fārsī, Ṭūsī, polyhedron, sphere, geometry.

Introduction

The treatise we are examining here, manuscript Tunis 16167/6 (73a/74a), is a commentary by Kamāl al-Dīn al-Fārsī (d. 1319) on some remarks occurring in Naṣīr al-Dīn al-Ṭūsī's *Taḥrīr uṣūl al-handasa* that concern the construction of semi-regular polyhedra inscribed into a sphere.

The Tunis codex starts with an indication that Fārsī intends to comment on the last paragraphs of the thirteenth *maqāla* of al-Ṭūsī's *Taḥrīr uṣūl uqlīdis* [Thirteenth Book of al-Ṭūsī's Exposition of Euclid's *Elements*][1] starting with the sentence " it is necessary that no more than two angles … " and finishing with the end of the book. Here is exactly what al-Ṭūsī writes:

> Even if it is not required that the faces of a solid belong to a single species, **it is necessary that no more than two angles** <ending at each of the vertices> be of the same kind, so that the solid does not lose its similarities <i.e. its symmetries> and hence cannot be inscribed within a sphere. Then, the number of solid angles <i.e. vertices> has to be even, exactly four, since two <angles> cannot constitute <a solid angle>, while six or more <vertices> would exceed four right angles. And the species of one of the faces has to be a triangle for the same reason. When <the faces of the polyhedron> are composed of triangles and squares, the figure has fourteen faces, eight triangles and six squares. It is as if it were composed of a cube and an octahedron and its side will be equal to the side of the hexagon occurring in the great circle of the sphere. When <the faces of the polyhedron> are triangles and pentagons, the figure has thirty-two faces, twenty triangles and twelve pentagons. It is as if it were composed of these two figures and its side will be equal to side of the decagon occurring in the great circle of the sphere. It results from this that the number of solids inscribed into a sphere is seven. We have

1. In this paper we refer to the second lithograph edition of al-±ū∞ī's *Exposition of the* Elements. Tehran 1880.

> ended the thirteenth *maqāla* which is the end of the book (*Taḥrīr*, 197). [1]

In these paragraphs, al-Ṭūsī shows that there are only seven polyhedrons which can be inscribed in a sphere, adding to the five regular Platonic solids two Archimedean ones: the cubo-octahedron (which has fourteen faces, eight equilateral triangles and six squares) and the icosi-dodecahedron (which has thirty-two faces, twenty triangles and twelve squares)[2].

Fārsī's intention here is to construct a representative example of a family of semi-regular polyhedra (the prisms) inscribed in a given sphere.

Fārsī's place in the Arabic Euclidean tradition

A brilliant student of Quṭb al-Dīn al-Shirāzī (d. 710/1310), Kamāl al-Dīn al-Fārsī (d. 718/1318) was one the great Persian specialist of optics, the author of *Tanqīḥ al-manādhir*, a critical commentary of Ibn al-Haytham's *Kitāb al-manādhir.* He is also known for his work on amicable numbers and his commentary on Ibn al-Khawwām's (d. 724/1324) *Al-Fawā'id al-bahā'iyya fī l-qawā'id al-ḥisābiyya*, an important text book in arithmetic, algebra and practical geometry. But, as far as we know, none of the standard bio-bliographical sources has credited him with any substantial work in Euclidean geometry; only some of his short commentaries and glosses in this field are extant. These are:

١. وإن لم يشترط أن يكون القواعد من جنس واحد **وجب أن لا يتجاوز فيه زاويتان** من جنس واحد لئلا يخرج الشكل من التشابه، فيمتنع وقوعه في الكرة. وحينئذ يكون الواقعة منها في الزاوية المجسمة عددًا زوجًا ، وهو أربعة لا غير لامتناع التأليف من اثنين وكون الستة و ما فوقها مجاوزة لأربع قوائم. ويجب أن يكون أحد الجنسين مثلثًا لئلا يتجاوز أيضا من ذلك. فإن كان التأليف من مثلثات ومربعات كان الشكل ذا أربعة عشر قواعد ثمان منها مثلثات وستة مربّعات ، كأنه مؤلف من المكعب وذي الثماني قواعد وضلعه يكون ضلع المسدس الواقع في أعظم دوائر الكرة. وإن كان من مثلثات ومخمسات كان ذا اثنتين وثلاثين قاعدة ، عشرين من المثلثات واثني عشرة من المخمسات. كأنه مؤلف من هذين الشكلين و ضلعه يكون ضلع المعشر الواقع في أعظم دوائر الكرة. ويصير بذلك المجسمات الواقعة في الكرة سبعة. تممنا المقالة الثلاثة عشر وهي آخر الكتاب (ص١٩٧).

2. De Young [2008, 181-191] shows that the anonymous author of Book XVI to Euclid's *Elements* has provided a construction of these two semi-regular polyhedrons and has proven them to be inscribed in a sphere. (Propositions: 7 to 11).

- *Risāla 'ala Taḥrīr al-Abharī fī l-mas'ala al-mashhūra min kitāb uqlīdis* (Treatise on Abharī's exposition on the well-known problem of Euclid's book).[1]
- *Risāla fī l-zāwiyya* (Treatise on angle).[2]
- *An-nadhar fī qawl al-Ṭūsī fī ākhir al-maqāla al-thālithah 'ashar* (Discussion of what al-Ṭūsī said at the end of Book XIII). This is the title we propose for the treatise we are presenting in this paper.

Fārsī's propositions

Constructing a prism inscribed in a given sphere

Fārsī's treatise is composed of two independent propositions. He places himself in the continuity of al-Ṭūsī's argumentation, introducing his additions by a simple sentence: "this <i.e. subject> needs to be discussed here"[3] and he does not seem to be compelled to add any theoretical arguments for justifying the new construction.

The first proposition is not formally enunciated; however it might be reconstructed in four parts: (1) Construction of a triangular prism inscribed in a sphere. (2) Construction of an equilateral triangular prism inscribed in a sphere. (3) Construction of any equilateral polygonal prism inscribed in a sphere. (4) Equilateral prisms inscribed in a sphere are limitless.

The first construction starts directly by its first step:

1. Draw a sphere and one of its diameters.
2. Cut the sphere with a plane perpendicular to the diameter so that the section is a small circle.
3. Draw an equilateral triangle inscribed into the circle.

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1. We have prepared an edition and a translation into English of this anonymous treatise which has been proposed for publication.
2. For two recent editions of this treatise, see Mawālidī [2014].

٣. في هذا القول نظر

4. On the sphere and on the other side of its diameter, draw a circle equal to the first circle.
5. Draw perpendiculars to the first circle from each vertex of the triangle. They end at the vertices on the circumference of the second circle and make a triangle equal to the first.
6. Join the vertices of the two triangles, each one to the one below it.

We get a right prism inscribed in a sphere with triangular bases and three rectangular faces. Fārsī does not conclude that he got what he wanted; on the contrary he adds here: “the height of the prism is greater than the side of the triangular base”.

For the second part, he offers a more precise construction:

Imagine now the movement toward the center <of the sphere> of the two cutting surfaces unchanged in their relation to the diameter, moving with similar movements, making the sides of the triangles greater and the sides of the rectangles smaller, until they all become equal.

Fārsī asks the reader to imagine a simultaneous movement of the two bases toward the center of the sphere. It makes the sides of the triangular bases greater and the sides of the rectangular faces smaller. The movement is stopped when the edges all become equal.

Now the conclusion is explicitly stated: “The produced prism has therefore equilateral faces; it is circumscribed by a sphere and each of its angles is made of three plane angles: two angles of a square and one of an <equilateral> triangle”.

The third part is a first porism: “The same method can be used for constructing pentagonal prism with five square lateral faces”.

The fourth part is another porism:” Solids with all edges equal and inscribed in a circle are limitless”.

It is quite remarkable that no diagrams are used in this proof and the procedure is not instantiated in any particular case.

The following diagram shows the final figure in modern perspective.

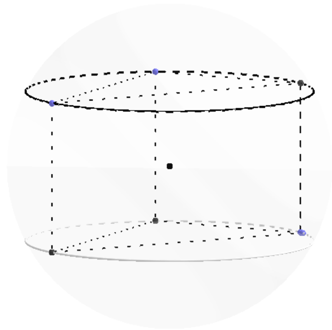

Figure 1. Imagining a movement of a plane

In order to get squares as lateral faces, Fārsī uses the movement as a way for the solution. For purist Euclidian geometers, motion was not allowed in demonstrations; however Fārsī is not the first Arabic mathematician to do it: before him, Thābit ibn Qurra introduced motion in proofs when he tried to demonstrate the parallel postulate. The same approach is found in al-Sijzī's *Kitāb fī tahsil al-subul lī istikhrāj al-ashkāl al-handasiyya.*[1] When they use motion in a proof, both authors introduce the verb "imagine". For example, Thābit writes: "When we imagine a solid moving in a unique direction according to uniform and rectilinear movement, any point of the solid describes a straight line" [2] and similarly, al-Sizjī writes: "Imagine a straight line moving …"[3]. Later when trying to prove the parallel postulate, Ibn al-Haytham introduces also the movement of the finite straight line perpendicular to a fixed line, the extremity of which describing a straight line parallel to the fixed one. Fārsī uses also motion in imagination and it was perhaps due to his familiarity with Ibn al-Haytham's works that he thought it suitable to follow his ideas and found an original way to construct a new class of polygonal prisms inscribed within a sphere unrelated to the classical techniques used in the Archimedean tradition.[4]

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1. These works are edited, commented on and translated into French in Rashed [2009, 688-931].

٢. كل مجسم نتوهمه متحركًا بكليته إلى جهة واحدة حركة واحدة بسيطة على استقامة فإن كل نقطة منه فهي تتحرك على استقامة.

٣. توهم خطًا متحركًا ...

4. 'Umar al-Khayyam rejected completely the introduction of motion in geometry (See his "Commentary on the Difficulties of Certain Postulates of Euclid's Book", in [Rashed R. and Vahabzadeh B. 2000, pp. 219-220]. Vitrac [2007] gives a complete analysis of the use of

Fārsī's second proposition

In this section of the treatise, Fārsī is more conventional: he enunciates the problem: To inscribe in a sphere a polygonal prism similar to a given polygonal prism. The construction is done in two phases. In the first phase, given the polygonal prism whose side is equal to the given length AB, he finds the sphere in which this polygonal prism can be inscribed. In the second phase, the polygonal prism similar to the above polygonal prism is inscribed in the given sphere.

The first phase can be divided into 9 steps:

Let AB be the circle in which a polygon to the side AB is inscribed, let Z be its center.

Take AD perpendicular to the plane of the circle and AD = AB. Let E be the midpoint of AD.

Let EH be parallel to AZ contained in the plane DA, AZ, and EH = AZ.

Then EH is collinear to the diameter of the sphere and EH is contained in the plane, D belongs to the surface of the sphere.

HZ is perpendicular on the surface of the circle as is AE.

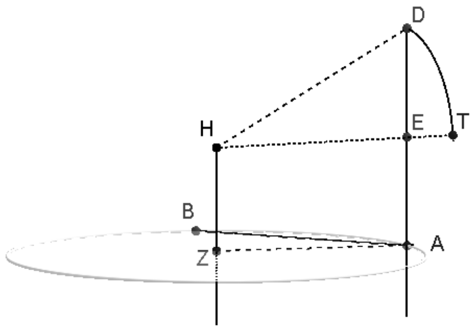

Figure2

Then H is the center of the sphere.

Then DH is the radius of the sphere and the diagonal of rectangular triangle DEH.

Extend EH by ET with T on the sphere

movement in the Euclidean geometrical tradition and an important bibliography on the subject.

Then HT = HD.

The second phase of the construction takes place in a great circle of the given sphere:

In the given sphere, draw great arc KSL with KL its diameter and M its center.

Let KM be cut at N so that the ratio of KN to NM is equal the ratio of TE to EH.

From N, draw the perpendicular NS to KL.

Then the arcs DT and SK are similar.

So the ratio of SN to NM is equal to the ratio of DE to EH.

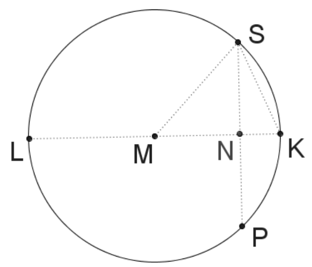

Figure 3

Let us show the similarity of the arcs DT and SK in modern notations.

KN : NM ∷ TE : EH (definition of the position of N on line KL)

KM : NM ∷ TH : EH (Porism of Euclid's Prop. V-19)

MS : NM ∷ HD : EH (since KM = MS and TH = HD as radius in their respective circles)

So the right triangles MNS and HED are similar and the interior angle KMS is equal the interior angle THD. This is equivalent to say that le great arc SK is similar to the great arc DT which also implies that SN : NM ∷ DE : EH.

We now continue the construction:

Construct trough S a plane perpendicular to SN.

It produces on the sphere a circle with a radius equal to NM.

Extend SN toward P such that P be on the circle.

Therefore, we have SP : NM ∷ DA : AZ , since SP = 2SN and DA = 2DE and EH = AZ (by specification).

Let SY be the edge of the constructed polygon inscribed in the circle produced by the plane passing through S on the given sphere. Then, since the radius of this circle is equal NM, so the ratio of the radius of the circle to the edge of the polygon, noted KM : SY, is the same as AZ : AB.

Then Fārsī concludes that, “ex aequali” (*bi’l musāwāt*) [1], the ratio of SP to SY is the same as the ratio of DA to AB. Indeed, he has proved that:

SP : KM ∷ DA : AZ and KM : SY ∷ AZ : AB, thus SP : SY ∷ DA : AB.

And since par specification, DA = AB, we have SY = SP. The edge produced is equal SP.

Once the base inscribed in a circle passing through the point P is similarly produced, we get the prism[2] as required.

It is clear that the first phase of the construction is intended at analyzing the geometrical characteristics of a sphere circumscribed to a given polygonal prism. Ultimately, the aim is to fix the ratio of the edge of the polygon to the radius of the circle. The second phase aims at constructing the specified prism and proving that all its edges are equal.

In this text, Fārsī uses with great mastery the Euclidian techniques, imagining figures in the space and manipulating proportions and he does not think necessary the insertion of explicit references to Euclid’s or to al-Ṭūsī’s *Recension of the Elements* in his elaborate proofs.

Discussions on semiregular polyhedra inscribed in a sphere

In the Arabic Euclidean tradition, Fārsī’s text is not the only attempt to construct semi-regular polyhedra inscribed in a sphere. There is at least another extant construction included in the so-called Book XVI, an anonymous addition to Euclid’s *Elements* presented by De Young[3]. This treatise contains

> “nineteen propositions describing techniques for constructing polyhedra within other polyhedra or within spheres. (…) The contents are, however, in the tradition of Archimedes rather that Euclid. (…) The addendum ends with what may be the earliest discussion of the construction of a representative example from each of the

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1. The expression “ex aequali” is based on Euclid’s V-22 and indicates an inference of the following kind:
If a : b ∷ d : e and b : c ∷ e : f then a : c ∷ d : f.
2. The word used by the author is “usṭuwānah” which literally means cylinder.
3. De Young [2008, 133-209] presents this Book XVI appended to a manuscript containing al-Ṭūsī’s *Tahrīr*. It is a unique copy dated 1593/4 and its author is unknown.

> classes of semiregular polyhedra known today as prisms and antiprisms." [De Young 2008, 133]

Among these propositions, the eighteenth describes the construction in a sphere of "a polyhedron having equilateral faces, two of which are specified figures occurring in a single circle and the remainder are squares" [De Young 2008, 200]. Therefore, the anonymous author of Book XVI discusses exactly the same problem as Fārsī, but they differ in their approaches as it appears from De Young's report:

"In this proposition, we construct a decagonal prism. The figure consists of two planes parallel to a great circle such that two planes cut the sphere forming equal circles. In these two circles we construct our desired equilateral figures – in this case decagons. We arrange these figures so that the vertices of one lie directly over the vertices of the other plane and connect the two vertices by lines between the two planes. We show that these connecting lines are perpendicular to the planes of the circles and that they are equal to one another." [De Young 2008, 201]

Let us summarize the steps of this construction:

Consider first a decagon inscribed in a circle.

Construct a rectangle WEZH with WE equal the diameter and EZ equal the edge of the decagon.

In a great circle of the given sphere, inscribe a rectangle ABGD similar to WEZH, with AD corresponding to WE.

Draw the two circles obtained by the intersection of the given sphere with the planes perpendicular in A and D to AD.

Draw in each of the two circles decagons and make their beginning points A and B.

Then we get a decagonal prism inscribed in the given sphere.

The following diagrams show the different steps in modern perspective:

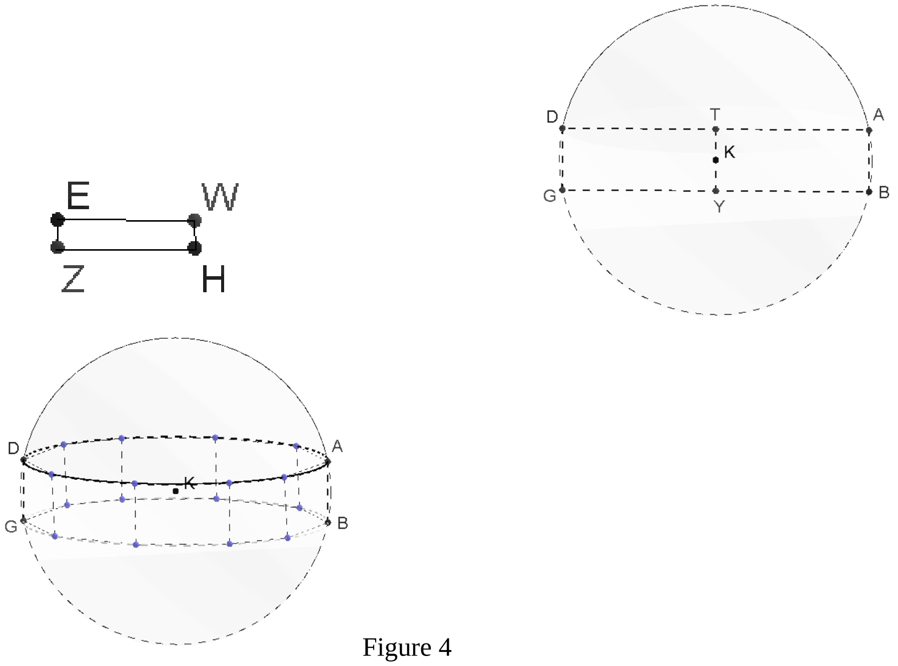

Figure 4

The anonymous author assumes implicitly two lemmas: the first is a direct consequence of Euclid’s proposition VI-4 concerning similar regular polygons inscribed in different circles; it says that for all of them, the ratio of the edge of the polygon to its diameter is the same. The second lemma is Proposition 17 of Book XVI that shows that it is always possible to draw inside a circle a quadrilateral similar to a given right-angled parallelogram.

Thus, the rectangle WEZH chosen at the beginning of the proof has the good dimensions: EZ is equal to the edge of a decagon inscribed in a circle that has a diameter equal to WE and the prism made at the end of the proof has also the good dimensions: AB is equal the edge of the octagon inscribed in the circles making the bases of the cylinder. [1]

Final remarks

Three types of constructions of right prisms inscribed in a given sphere, with different proofs, have been presented here; Fārsī’s first one is highly interesting since it uses motion but it proves the

1. De Young [2008, 199-202 and 165-166] presents an Arabic edition of the propositions 17 and 18 of Book XVI and their translation into English with commentaries and notes.

existence of these objects. Like the anonymous proof, his second proof is more traditional, taking its inspiration and techniques in the Euclidean tradition, but we have shown that the two are different in their approaches.

In his paper, De Young presents the history of regular and semiregular polyhedra and asserts that Propositions 18 and 19 "may be the earliest discussion of the construction of a representative example from each of the classes of semiregular polyhedra known today as prisms and antiprisms" [De Young 2008, 133]. We do not know if Fārsī had read the anonymous Book XVI and attempted to add his own constructions or if he did not read it. Anyhow, this treatise confirms that he was an exceptional mathematician.

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The manuscript Tunis Mss-16167/6

The short work under consideration here belongs to the codex Tunis Mss-16167 (also known as Ahmadiyya 8452) and is the sixth unit (73a-74a) among ten all devoted to commentaries on Euclid's Elements.[1] Rashed [2002, 736] presents a short description of the codex but ignores the existence of this particular Fārsī's treatise. We discovered another copy of this treatise: the fourteenth unit of the codex Leiden Or.14, copied around 1667. After examining the two manuscripts[2], it appears that the Leiden manuscript is a copy of the Tunis one[3], but it contains significant scribal errors that make it unsuitable for the edition of the text. Modaras Radwy [1975] indicates that there is another copy of this treatise in Mashhad (Iran), however we did not get a copy of it.

The Tunis codex is composed of 90 folios, 13x21,5 cm, 23 lines each and with *nasta'liq* script, and has been written by a unique copyist: Darwīsh Ahmad al-Karīmi who ended copying it in 869/1464. Fārsī's treatise contains only two diagrams placed in a unique rectangular "window", but the first diagram is difficult to read for it encroaches on the text and letters of the diagram are mixed with those of the text. The Leiden copy was no help since the "windows" stayed empty of diagrams; however, we used it in several instances in order to remove a doubt concerning ambiguous words or, if needed, to correct and adjust the meaning of a sentence.

1. This volume also contains the well known Ibn al-Haytham's (d. 1038) *Sharḥ musādarāt uqlīdis l-Ibn al-Haytham* [Commentary on the Premises of Euclid's Elements] (ff. 1b-59b), Al-'Abbās ibn Sa'īd al-Jawharī's (d. 835) *Ziyādāt al-'Abbās ibn Sa'īd fī l-maqāla al-khāmisa min uqlīdis* [Additions to the Fifth Book of Euclid's *Elements*] (ff. 60b-61a) and Thābit b. Qurra's (d. 901) *Fi l-'illati l-lati lahā rattaba uqlīdis ashkāl kitābihi dhālika l-tartībi* [Treatise on the Cause of why Euclid disposed Propositions of his book in such order] (ff. 86b-90b). Most of the treatises of this collection of manuscripts have been analyzed; some have even been edited and translated into French, English or Persian.
2. I obtained a copy of Leiden 14/14 thanks to Professor Pierre Ageron (University of Caen).
3. Rashed [2002, 737] shows that four other units (18-19-20-21) of the codex Leiden 14 have also been copied directly from the Tunisian codex.

Edition and translation of MSS_16167-6

He said: In the name of Allah, the most merciful, the most gracious. The greatest, the supreme, the guide of the greatest sages, the chief of a cohort of scientists, the completeness in the state and the religion, al-Hasan al-Fārsī, may God receives him in his garden and gives him its fresh water, said: what The sage knowledgably man of science, Nā∞ir Al-Dawla wa l-Dīn, said at the end of Book XIII: "It cannot exceed two right angles", to the end of the phrase, needs to be discussed.

We draw the sphere and one of its diameters. We imagine a plane surface perpendicular to this diameter and cutting the sphere; it produces a very small circle, as for example a circle with a diameter of 10 while the diameter of the sphere is 120 degrees. <We draw> in this circle an equilateral triangle and in the other side of the diameter an identical circle. Then, from the vertices of the triangle we draw three perpendiculars to its surface. They end on the circumference of the second circle. Joining the three ends we get another triangle equal to the first. From the three perpendiculars and the sides of the two triangles, three equal rectangles are produced. The five surfaces produced are the faces of a prism which is such that its height is greater than the sides of its base.

Imagine now the movement toward the center <of the sphere> of the two cutting surfaces unchanged in their relation to the diameter, moving with similar movements, making the sides of the triangles greater and the sides of the rectangles smaller, until they all become equals.

The produced prism has therefore as bases equilateral triangles, it is inscribed into a sphere and each of its angles are made of three plane angles two of them are angles of a square and one of them is the angle of a triangle. The same can be said concerning a cylinder with bases which are pentagonal or any other regular polygon with equal angles and sides. Since every angle of a polygon is smaller than two right angles, it can produce with two right angles belonging to a rectangle always a solid angle. Therefore, the kinds of solids with all edges equal and inscribed in a sphere can be indefinitely great. That is what we wanted.

قال بسم اللّه الرحمن الرحيم، قال المولى المعظم قدوة أكابر الحكماء ورئيس فحول العلماء كمال الدولة والدين الحسن الفارسي أسكنه الله تعالى جنانه وسقاه مياهه: إن ما قاله الحكيم المحقق والحبر المدقق نصير الدولة والدين في آخر المقالة الثالثة عشر «وجب أن لا يتجاوز فيه زاويتان ...» إلى آخره، في هذا القول نظر .

وذلك أن نرسم الكرة وقطرا من اقطارها ونتوهم سطحا مستويا يقوم عليه القطر ويقطع الكرة فيحدث دائرة صغيرة جدا قطرها عاشرة مثلا على ان قطر الكرة مائة وعشرون درجة وفيها مثلث متساوي الاضلاع ودائرة مثلها في الطرف الآخر من القطر. ثم نخرج من زوايا المثلث ثلاثة اعمدة على سطحه فتنتهي إلى محيط الدائرة الثانية. فإذا وصلنا بين أطرافها حدث مثلث آخر مثل الأول سواء. ويحدث من الاعمدة الثلاثة وأضلاع المثلثين ثلاثة مستطيلات متساوية. ويحيط السطوح الخمسة بمنشور ويكون ارتفاعه أعظم من ضلع قاعدته.

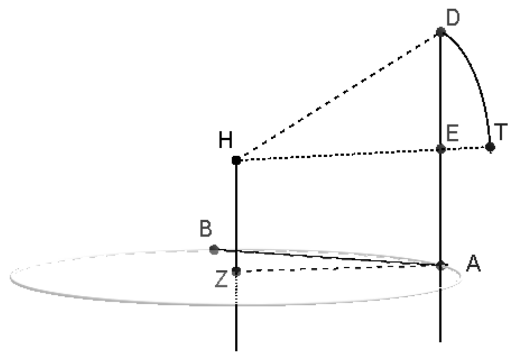
فإذا توهمنا حركة السطحين القاطعين نحو المركز على وضعهما مع القطر حركتين متساويتين تعاظمت أضلاع المثلثين وتصاغرت أطوال المستطيلات إلى أن يتساوى جميعا.

فيكون هذا المنشور ذا قواعد مساوية الاضلاع ويحيط به كرة وتركب كل زاوية من زواياه من ثلاث زوايا مسطحة اثنان منها زاويتا مربع وواحدة منها زاوية مثلث. وكذلك القول في اسطوانات قواعدها مخمس أو غيرها من الأشكال الكثيرة الأضلاع المتساوية الزوايا والأضلاع. ولأن كل زاوية من زوايا الكثيرة الأضلاع أقل من قائمتين أبدًا فيمكن أن يحيط مع قائمتين من مربّعين بزاوية مجسمة. فإذا انواع المجسمات المتساوية أضلاع القواعد التي تحيط بها الكرة لا يتناهى كبرها. وذلك ما أردناه.

Then when we want to construct one of the above mentioned solids in a sphere, we draw a circle and construct in it a figure similar to the base of the solid. Let AB be the circle, Z its center and AB a side of the polygon inscribed in it. Join AZ and take AD <perpendicular to the circle> and equal to AB. Let it be bisected at E. From E draw the parallel EH to AZ contained in the plane DA, AR. Let EH = AZ.

It is clear that if the circle AB where one of the two bases of a cylinder inscribed in a sphere, EH would be collinear to the diameter of the sphere, because the plane parallel to the base and bisecting the cylinder would necessarily pass through the center of the sphere, the two circles of the bases being equal. Therefore EH is contained in this plane and D belongs to the surface of the sphere. Join HZ, which is perpendicular on the surface of the circle as is AE.

Then H is the center of the sphere. Join DH; it is the radius of the sphere and (…) on DE and EH. We extend EH by HT equal to HD and we draw the arc DT of a great circle of the sphere with H as a center and HD as a radius.

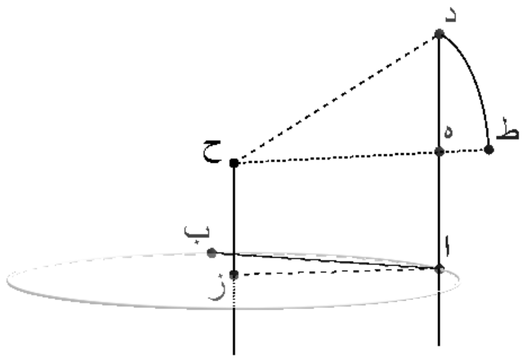


Then we first draw on the given sphere the great circle KSL with KL as its diameter and M as its center. Let KM be cut at N so that the ratio of KN to NM be the same as the ratio of TE to EH. And from N <we draw> the perpendicular NS and we join SM and SK.

ثم إذا اريد عمل مجسم من المذكورات في كرة رسمت دائرة وعمل فيها شكل يشبه قاعدة المجسم. ولتكن الدائرة اب ومركزها ز وضلع الكثير الاضلاع المعمول فيها اب ونصل از ونخرج اد مثل اب وننصفه على ه ونخرج من ه في سطح خطي دا ، از (خط) هح موازيًا (لخط) از ونجعل هح‌ا‌از.

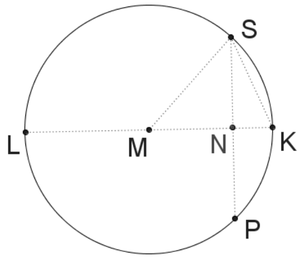
فبيّن ان دائرة اب إذا كانت احدى دائرتي قاعدة الاسطوانة في كرة كان هح على استقامة قطر الكرة لان السطح المنصّف للاسطوانة على موازاة القاعدة يمر بمركز الكرة ضرورة كون دائرتي قاعدتها متساويتين. ويكون هح في ذلك السطح ونقطة د على سطح الكرة. ونصل حز وهو عمود على سطح الدائرة ومثل ها.

ف ح مركز الكرة ونصل دح، فهو نصف قطر الكرة، (...)[1] على ده، هح. ونخرج هح إلى ط مساويًا لـ حد ونرسم على ح ببعد حد قوس دط من عظيمة تلك الكرة.

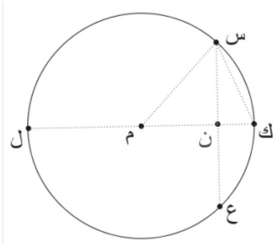


ثم نرسم في الكرة المفروضة أولا عظيمة كـس‌ل و كـل قطرها والمركز م. ونقسم كـم على ن حتى يكون نسبة كـن إلى نم كنسبة طه إلى هح. ومن ن عمود ن‌س ونصل س‌م، س‌ک.

١. الكلمة غير مفهومة.



Since the ratio of KN to KM is the same as the ratio of TE to TH, then the arcs DT and SK are similar and the ratio of SN to NM is the same as the ratio of DE to EH. We construct through S a plane perpendicular to SN; it produces a circle with a radius equal NM. We extend SN toward P; then the ratio of SP to its radius is the same as the ratio of DA to AZ and the ratio of its radius to the edge of the figure, constructed in the sphere, that has a base similar to the base of the cylinder is the same as the ratio of ZA to AB. Then *ex aequalia*, the ratio of the BP to the edge of the figure constructed in the sphere is as the ratio of DA to AB. Then the constructed edge is equal to SP. The same can be said for the base inscribed in a circle passing through the point P, and once the figure drawn, we get the cylinder as posed. That is what we wanted.



فلأن نسبة كـن إلى كـم كنسبة طه إلى طح، فقوس دط شبه قوس سك ونسبة سن إلى نم كنسبة ده إلى هح فنخرج سطحًا على س يكون سن عمودًا عليه. فيحدث دائرة، نصف قطرها يساوي نم ونخرج سن إلى ع، فيكون نسبة سع إلى نصف قطرها كنسبة دا إلى از ونسبة نصف قطرها إلى ضلع الشكل المعمول فيها الشبيه بقاعدة الاسطوانة كنسبة از إلى اب. فبالمساواة نسبة بع إلى ضلع الشكل المعمول فيها كنسبة دا إلى اب. فالضلع المعمول هو بقدر سع. وكذلك القول في القاعدة الأخرى المعمولة في دائرة تمر بنقطة ع فإذا أتممنا الشكل حصل اسطوانة كما فرضت وذلك ما أردناه.

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