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The Five Arabic Revisions of Autolycus' *On the Moving Sphere* (Proposition VII)

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Abstract

Autolycus' *On the Moving Sphere* is among the earliest Greek mathematical works to reach us and was considered a staple of the middle books which were intended to be studied between the *Elements* and the *Almagest*. This text was also among the many mathematical works translated to Arabic in the 3rd/9th century and was the subject of several translations, redactions, and recensions. In analyzing the available copies of *al-Kura al-mutaḥarrika*, we identified five different versions. In this paper, we study these five versions, with a focus on the seventh proposition, in order to characterize their relations to each other and to the extant Greek text.

Keywords: Autolycus (Ūṭūlūqus), Isḥāq ibn Ḥunayn, al-Kindī, Middle books, Muḥyī al-Dīn al-Maghribī, Naṣīr al-Dīn al-Ṭūsī, *On the Moving Sphere* (*Fī al-kura al-mutaḥarrika*), Thābit ibn Qurra.

Introduction

Autolycus of Pitane (*fl. c.* 300 BC) was a successor to Eudoxus (d. *c*. 347 BC) in the study of spherical astronomy and was perhaps a predecessor or contemporary of Euclid (*fl. c.* 295 BC). The two treatises of Autolycus, Περὶ κινουμένης σφαίρας (*De sphaera quae movetur, al-Kura al-mutaḥarrika*, *On the Moving Sphere*) and Περὶ ἐπιτολῶν καὶ δύσεων (*De oritubus et occasibus*, *Fī al-tulūʿ wa-al-ghurūb*, *On Risings and Settings*), are among the earliest Greek astronomical works to survive in their entirety. It is commonly held that *On the Moving Sphere* is earlier than Euclid's *Phaenomena*, on the grounds that Euclid appears to make use of Autolycus in his own work.[1] As Heath put it, "That [Autolycus] wrote earlier than Euclid is clear from the fact that Euclid, in his similar work, the *Phaenomena*, makes use of propositions appearing in Autolycus, though, as usual in such cases, giving no indication of their source."[2] However, Neugebauer found this argument for the relative dating of Autolycus and Euclid to be "singularly naïve" and not strong enough to rule out the possibility that the two were contemporary.[3] Dating Autolycus is important since, as we will see later, the form and method of proof in Autolycus is very similar to what we find in Euclid's *Elements*.

1. See, for example, Germaine Aujac, *Autolycos de Pitane. La sphère en mouvement. Levers et couchers héliaques, testimonia* (Paris: Les Belles Lettres, 1979), 8–10; G. L. Huxley, "Autolycus of Pitane," in *Dictionary of Scientific Biography*, ed. Charles Coulston Gillispie (New York: Charles Scribner's Sons, 1970), 1:338–39; Joseph Mogenet, *Autolycus de Pitane: histoire du texte suivie de l'édition critique des traités De la sphère en mouvement et Des levers et couchers* (Louvain: Bibliothèque de l'Université, Bureaux du recueil, 1950), 5–9; Paul Tannery, *Mémoires Scientifiques*, ed. H. G. Zeuthen and J. L. Heiberg, vol. II: Sciences exactes dans l'antiquité (Toulouse: Édouard Privat, 1912), 225.
2. Thomas Little Heath, *A History of Greek Mathematics: From Thales to Euclid*, vol. 1 (Oxford: Clarendon Press, 1921), 349.
3. Otto Neugebauer, *A History of Ancient Mathematical Astronomy* (Berlin; New York: Springer-Verlag, 1975), 750. See also: Alan C. Bowen, "Autolycus of Pitane," in *The Encyclopedia of Ancient History*, ed. Roger S. Bagnall, Kai Brodersen, Craige Brian Champion, Andrew Erskine, and Sabine R. Huebner (Malden, MA: Wiley-Blackwell, 2013), 2:980–81; Alan C. Bowen and Bernard R. Goldstein, "Hipparchus' Treatment of Early Greek Astronomy: The Case of Eudoxus and the Length of Daytime," *Proceedings of the American Philosophical Society* 135, no. 2 (1991): 246. Bowen and Goldstein build on Neugebauer's refutation of Heath to argue that the usual arguments for dating Autolycus to before Euclid are equally weak: "the assumption that Autolycus must have lived in the fourth century seems ultimately based on the dubious assumption that attention to simpler and less sophisticated theories generally belongs to a period prior to that of concern with more complex and sophisticated theories."

Many scholars have suggested that both Euclid and Autolycus made use of an earlier work on the sphere, unknown to us, from which they appropriated several propositions without themselves providing the proofs. In the course of proving proposition 7 of *On the Moving Sphere,* Autolycus uses, without explicit proof, what would later become proposition III.1 of Theodosius' *Sphaerica* (d. *c*. 399 BC) and to which we will return later in this paper. Although it is not clear if Euclid used Autolycus' work or if they were both using a similar source, it has been generally accepted that the method of proof used in their works had a long history before them.

It is well known that the Greek method of proof was adapted into medieval Arabic and Persian mathematical works, but the nature of this adaptation and the possible changes resulting from the appropriation requires further research. One approach to studying the adaptation of this style of proof into Arabic is to examine the various Arabic translations and recensions of Greek mathematical works against the extant Greek texts. In the course of the 2–4th/8–10th century translation movement, virtually all Greek mathematical works were translated into Arabic and, interestingly, many of these texts were translated several times. In addition to this practice of re-translation, it was common for different scholars to prepare their own recensions of the early translations. Autolycus' *On the Moving Sphere* was one text in this tradition, subject to translation, re-translation, and the production of recensions.[١]

In analyzing the available Arabic copies of Autolycus' *On the Moving Sphere*, we were able to distinguish five different versions, outlined in the following table:[٢]

1. Another example is the *Sphaerica* of Theodosius, about which see: Richard Lorch, "The 'Second' Arabic Translation of Theodosius' *Sphaerica*," in *From Alexandria, Through Baghdad: Surveys and Studies in the Ancient Greek and Medieval Islamic Mathematical Sciences in Honor of J.L. Berggren*, ed. Nathan Sidoli and Glen Van Brummelen (Berlin; Heidelberg: Springer-Verlag, 2014), 255–58; Paul Kunitzsch and Richard Lorch, *Theodosius Sphaerica: Arabic and Medieval Latin Translations* (Stuttgart: Steiner, 2010).
2. For each version, we have indicated all the copies that we could locate, except for version IV (Ṭūsī's recension) for which we have listed only a selection of the most important witnesses.

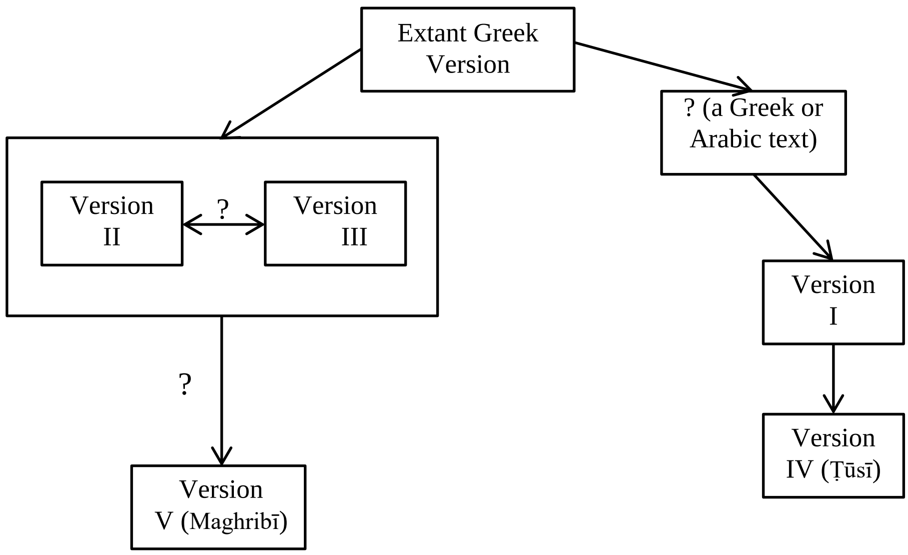
Version	Sigla	Manuscript
I	A (آ)	Turkey, Istanbul, Topkapı Saray Müzesi Kütüphanesi, Ahmet III, 3464, ff. 54b–58b (early 7th/13th century).
	K (ك)	Private Collection (formerly in possession of Paul Kraus), ff. 65b–70b.
	X (خ)	United Kingdom, Oxford, Bodleian Library, Huntington, 237, ff. 76a–82a.
	B (ب)	Iran, Tehran, Dānishgāh-i Tihrān, Kitābkhāna-yi Markazī, 1063, ff. 27a–28a.
II	F (ف)	Turkey, Istanbul, Köprülü Kütüphanesi, Fazıl Ahmed Paşa, 932, ff. 66a–66b (7th/13th century).
III	S (س)	Turkey, Istanbul, Süleymaniye Kütüphanesi, Ayasofya, 2671, ff. 122b–132a (copied 621/1224).
	L (ل)	United Kingdom, London, Institute of Ismaili Studies, Hamdani Collection, 1647, ff. 39b–40b, 2a-b, 13a-b, 41a-42b (7th/13th century).
IV (Ṭūsī)	M (م)	Iran, Tabriz, Kitābkhāna-yi Millī-yi Tabrīz, 3484, pp. 58–63 (7th/13th century).
	H (ح)	Turkey, Istanbul, Hacı Selim Ağa Kütüphanesi, Hacı Selim Ağa, 743, ff. 241b–243a (672/1274).
	G (ج)	Iran, Tehran, Sipahsālār (Muṭaharrī), 4727, ff. 67b-69b (17 Shaʿbān 671/9-10 March 1273).
V (Maghribī)	D (د)	Ireland, Dublin, Chester Beatty Library, Arabic, 3035, ff. 144a–147b (copied 669/1271).
	R (ر)	Iran, Tehran, Kitābkhāna-yi Majlis-i Shurā-yi Islāmī, Shurā, 200, ff. 254b–260a.
	Q (ق)	Iran, Mashhad, Kitābkhāna-yi Markazī Astān-i Quds, 5232, pp. 77–91 (7th/13th century).

One of the main tasks of this research project is to shed light on the authorial attribution of each of these versions. Based on a survey of the content of each version, compared to the Greek text, we can make several statements about the relation of the versions to each other which we will simply mention here and discuss more thoroughly as the paper proceeds.[١]

Versions II and III are very close to the extant Greek text and, most probably, one of these versions was prepared based on the other, but we have not come to a definitive conclusion on which is which. It is

1. See also: Francis J. Carmody, *The Astronomical Works of Thabit b. Qurra* (Berkeley: University of California Press, 1960), 217. Since the main focus of Carmody's work is not the recensions and translations of *al-Kura al-mutaḥarrika*, his statement on the topic is a bit ambiguous but he seems to be saying that the translation he attributed to Thābit (our version III) and the one he attributed to Isḥāq (our version I) are more similar to the Greek text than the recension of Ṭūsī.

important to note that we find in both of these versions passages which are not found in the extant Greek text. Version I, by contrast, seems to have been prepared either based on an unknown Greek recension or on a different Arabic translation of the extant Greek text. Version IV, Naṣīr al-Dīn al-Ṭūsī's (d. 1274) recension which would become the most widely circulated version, seems to have been adapted directly or indirectly from version I. Finally, version V is a recension prepared by Muḥyī al-Dīn Ibn Abī al-Shukr al-Maghribī (d. 1283) and, in comparison to versions I and IV, is more similar in content to the Greek text but is still notably different from versions II and III.[1] Moreover, there is strong evidence to deny any substantial dependence of Maghribī's version on that of Ṭūsī.[2]



1. See: Muḥsin ibn ʿAlī ibn Muḥammad Āghā Buzurg al-Ṭihrānī, *al-Dharīʿa ilá taṣānīf al-shīʿa*, vol. 3 (Beirut: Dār iḥyāʾ al-turāth al-ʿarabī, 2009), 383.

2. In addition to his recension of Autolycus' *al-Kura al-mutaḥarrika*, Maghribī is known to have prepared several recensions of Greek mathematical works for which Ṭūsī had already prepared his own recension and to which Maghribī may have had access. Examples of such works include Ptolemy's *Almagest*, Euclid's *Elements*, and Theodosius' *Sphaerica*. To what extent Maghribī's recensions depended on or relate to Ṭūsī's recensions remains an open question for research. See: Masoumeh Amirimoqaddam, "Al-Ṭūsī's Recension Method in Comparison to the Maghribī's Recension of Theodosius' Sphaerica," (in Persian) *Tarikh-e Elm: The Iranian Journal for the History of Science* 11, no. 1 (2013/1393): 1–30.

The authorship information gleaned from the manuscript copies of the five versions is not consistent. Among the five versions, the authorship of versions I, II, and III are unclear.

Ibn al-Nadīm (d. *c.* 995) only mentioned one person in relation to the Arabic translation of Autolycus' *al-Kura al-mutaḥarrika*: Abū Yūsuf Yaʿqūb Ibn Isḥāq al-Kindī (d. 873). Under the entry for Autolycus, he listed both *al-Kura al-mutaḥarrika* and *Fī al-ṭulūʿ wa-al-ghurūb*. Ibn al-Nadīm tells us that the former was corrected by Kindī (*iṣlāḥ al-Kindī*)[1] but Kindī's name is not mentioned in any of the copies we analyzed. The word *iṣlāḥ* (correction, modification, improvement) refers to a common practice in the 2–4th/8–10th century translation movement whereby one person was responsible for the initial translation of a text and another for correcting, rephrasing, and restructuring the text.

Kindī has, in fact, been credited with both the translation and correction of various mathematical works.[2] In the introduction to his recension of the first book of Ptolemy's *Almagest*, entitled *Kitāb fī al-ṣināʿa al-ʿuẓmá* (The greatest craft), Kindī refers the reader to two of his own works which may be related to the present work of Autolycus: *Kitāb fī al-kura* (On the sphere) and *Kitāb fī ḥarakat al-kura* (On the motion of the sphere), neither of which are extant.[3] Elsewhere in the body of this text, Kindī makes reference to a work of his entitled *Kitāb al-ukar* (On the spheres). We are not aware of any copies of this work

1. See: Ibn al-Nadīm, *Kitāb al-fihrist*, ed. Reza Tajaddod (Tehran: Marvi Offset Printing, 1393), 328. See also the edition by Ayman Fuʾād Sayyid (London: al-Furqan Islamic Heritage Foundation, 2009), 216. Ibn al-Nadīm did not mention this work under his entry for Kindī but he did list his *risāla fī al-kuriyyāt* (The treatise on the spheres) which may be a reference to Autolycus' treatise. Al-Qifṭī adds nothing to Ibn al-Nadīm. Ḥājī Khalīfa adds that the translation was first produced during the period of the Abbasid Caliph al-Maʾmūn and later corrected by Kindī, see: Ḥājī Khalīfa, *Kashf al-ẓunūn ʿan asāmī al-kutub wa-al-funūn*, ed. Şerefettin Yaltkaya and Kilisli Rifat Bilge, vol. 1 (Istanbul: Wikālat al-maʿārif, 1971), 142.
2. Franz Rosenthal, "Al-Kindī and Ptolemy," in *Studi orientalistici in onore di Giorgio Levi Della Vida*, vol. 2, Pubblicazioni dell'Istituto per l'Oriente 52 (Roma: Istituto per l'Oriente, 1956), 436–56; A.I. Sabra, "Some Remarks on al-Kindī as a Founder of Arabic Science and Philosophy," in *Dr. Mohammad Abdulhādi Abū Ridah Festschrift*, ed. ʿAbd Allāh ʿUmar (Kuwait: Kuwait University, 1993).
3. Franz Rosenthal, *Yaʿqūb Ibn Isḥāq al-Kindī: Fī al-ṣināʿat al-ʿuẓmá* (Nicosia, Cyprus: Dār al-Shabāb, 1987), 120.

but, from the context of Kindī's references, it does not seem relevant to Autolycus' *al-Kura al-mutaḥarrika*.[1]

Thābit ibn Qurra (d. 901) was also known for correcting the translations of many mathematical works, such as Euclid's *Element* and Ptolemy's *Almagest*. He apparently corrected Autolycus' text as well since Ṭūsī starts his recension of *On the Moving Sphere* with the words "*taḥrīru kitābi al-kurati al-mutaḥarrikati li-Ūṭūlūqusa aṣlaḥahū Thābit*" (the recension of the moving spheres by Autolycus corrected by Thābit [ibn Qurra]). This statement suggests that Autolycus' book was perhaps translated by someone else and later corrected by Thābit. On the other hand, one of the copies of version III (MS S) starts with "*kitābu Ūṭūlūqusa fī al-kurati al-mutaḥarrikati tarjamatu Thābiti ibni Qurrati wa-taṣḥīḥuhū*" (the book of Autolycus on the moving sphere, translated and corrected by Thābit ibn Qurra) and ends with, "*tamma kitābu Ūṭūlūqusa fī al-kurati al-mutaḥarrikati tarjamatu Thābiti ibni Qurrati al-Ḥarrānī*" (thus ends the book of Autolycus on the moving sphere, translated by Thābit ibn Qurra al-Ḥarranī). The second copy of the version III (MS L) ends with "*tamma kitābu Ūṭūlūqūsa fī al-kurati al-mutaḥarrikati iṣlāḥu Thābiti ibni Qurrati al-Ḥarrānī*" (thus ends the book of Autolycus on the moving sphere, corrected by Thābit ibn Qurra al-Ḥarranī). One may conclude that Ṭūsī was referring to this version III; however, as noted above and as will be discussed later in this paper, this is questionable since Ṭūsī's revision is actually based on version I, the copies of which do not bear Thābit's name.[2] At the end of MS A (version I), we find "*tamma kitābu Ūṭūlūqusa fī al-kurati al-mutaḥarrikati ikhrāju Isḥāqi ibni al-Ḥasani*" (thus ends the book of Autolycus on the moving spheres composed by Isḥāq ibn al-Ḥasan). The name "Isḥāq ibn al-Ḥasan" is probably due to a copyist mistake in writing "Isḥāq ibn Ḥunayn," a famous 3rd/9th century translator from Greek to Arabic. Yet, at the beginning of another copy of version I, MS X, Ḥunayn ibn Isḥāq (d. 873) is introduced as the translator of Autolycus' text. Ḥunayn ibn Isḥāq, the father of the former Isḥāq ibn Ḥunayn, is generally known for his translation of medical works. Other

1. Ibid., 175–76.
2. See: Younes Karamati, "Taḥrīr," in *Dāʾirat al-maʿārif-i buzurg-i Islāmī*, ed. Kāẓim Mūsavī Bujnūrdī (Tehran: Markaz-i dāʾirat al-maʿārif-i buzurg-i Islāmī, 2007/1385), 603–6. Karamati comes to the same conclusion as we do, *i.e.*, that Ṭūsī's version is most closely related to version I which bears the name of Isḥāq.

than searching for additional authentic copies of these versions, one way that we can further determine their authorship is to try to identify relationships between these versions in content, terminology, and structure. Thus, in the following sections of this paper, after briefly reviewing the printed editions of Autolycus' *On the Moving Sphere* and the content of the treatise, we compare the five Arabic versions and try to highlight the connections between these different versions.

The complete text of both works of Autolycus was first made available in a modern edition based on five manuscripts by Fridericus Hultsch with his own Latin translation in 1885.[1] Joseph Mogenet published a new critical edition of the Greek text in 1950, based on all forty-three manuscripts known at the time.[2] Mogenet had also published, a couple years earlier, an edition of the Latin translation, *De sphaera quae movetur*, by Gerard of Cremona (d. *c.* 1187).[3] In 1971, English translations of both works of Autolycus were published by Frans Bruin and Alexander Vondjidis, together with a re-printing of Hultsch's Greek editions.[4] The Greek texts were edited yet a third time and published in 1979 with French translations by Germaine Aujac based on the oldest and most precise manuscript witnesses.[5] The Arabic text of Ṭūsī's recension of *al-Kura al-mutaḥarrika* was published in Hyderabad in 1939.[6] None of the other Arabic versions studied in this paper have yet been published. Throughout the present study, we take the text prepared by Aujac to be representative of the Greek text of Autolycus and the most authoritative of the three editions of the Greek.

Since this flurry of scholarship, very little has been published on

1. Fridericus Hultsch, *Autolyci de sphaera quae movetur liber. De oritubus et occasibus libri duo* (Lipsiae: B.G. Teubner, 1885).
2. Joseph Mogenet, *Autolycus de Pitane: histoire du texte suivie de l'édition critique des traités De la sphère en mouvement et Des levers et couchers* (Louvain: Bibliothèque de l'Université, Bureaux du recueil, 1950).
3. Joseph Mogenet, "La traduction latine par Gérard de Crémone du Traité de la Sphère en Mouvement d'Autolycus," *Archives internationales d'histoire des sciences* 5 (1948): 139–64.
4. Frans Bruin and Alexander Vondjidis, *The Books of Autolykos: On a Moving Sphere and on Risings and Settings* (Beirut: American University of Beirut, 1971).
5. Aujac, *Autolycos de Pitane. La sphère en mouvement. Levers et couchers héliaques, testimonia* (Paris: Les Belles Lettres, 1979).
6. Naṣīr al-Dīn Muḥammad ibn Muḥammad al-Ṭūsī, *Majmūʿ al-rasāʾil*, 2 vols. (Hyderabad: Matbaʿat dāʾirat al-maʿārif al-ʿuthmāniyya, 1939).

Autolycus and next to nothing has been said of the Arabic translations and recensions, despite the fact that Ṭūsī's would circulate widely and become a staple of the works known as the *mutawassitāt*—mathematical texts intended to be read between Euclid's *Elements* and Ptolemy's *Almagest*. In 1900, Heinrich Suter listed two Arabic versions of Autolycus' *On the Moving Sphere*: a translation under the name of Isḥāq ibn Ḥunayn known to him in a single witness, MS X above, and several witnesses of Ṭūsī's recension.[1] Suter makes no comment on the veracity of the attribution of the former to Isḥāq or its relation to the latter version of Ṭūsī. To the best of our knowledge, the first modern scholar to recognize that there are indeed distinct Arabic versions of *al-Kura al-mutaḥarrika* that predate Ṭūsī's recension was Max Krause.[2] Krause listed two Arabic translations: one under the name Autolycus, our MS A (version I) which, as previously discussed, is attributed to Isḥāq ibn al-Ḥasan [*sic*] in the colophon and a second translation under the name of Thābit ibn Qurra, our MS S (version III). Krause also listed several witnesses of Ṭūsī's recension.[3] Fuat Sezgin, under his entry for Autolycus, listed three copies: our MSS A, S, and K, the latter having been previously held by Paul Kraus and now in a private collection.[4] Sezgin repeats the usual statment that the initial translation was done by Isḥāq and corrected by Thābit, however, he does not cite any manuscript witnesses under his entries for either figure. None of the aforementioned scholars have carefully examined the content of these various Arabic versions or situated them relative to the Greek text. In this paper, we present a preliminary such study, focusing on the seventh proposition of the work, placing it in the context of the whole text, and examining how it differs from one Arabic version to another.

On the Moving Sphere examines the celestial sphere as it rotates in

1. Heinrich Suter, *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig: B.G. Teubner, 1900), 40 and 152.
2. Max Krause, "Stambuler Handschriften islamischer Mathematiker," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*. Abteilung B, Studien 3 (1936): 440, 457, and 502.
3. See also: Karamati, "Taḥrīr." Karamati studied the introductions of the three copies cited by Krause to show that each represents a distinct translation or recension of the text of Autolycus.
4. Fuat Sezgin, *Geschichte des arabischen Schrifttums*, vol. 5 (Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften an der Johann Wolfgang Goethe-Universität, 1974), 82.

relation to a fixed horizon about an axis extending from pole to pole. The work opens with four preliminary assertions. It should be noted that in the Greek manuscripts no distinction is made between these first four statements and the main propositions of the text, nor are the former explicitly labeled as *ὅροι*.[1] Here, Autolycus states what he intends by uniform motion, the axis of a sphere, and the poles of a sphere. Following the first statement, a more involved assertion is made, without proof, that the ratio of the distances traversed by a point under uniform motion will equal the ratio of the corresponding time intervals.

Propositions 1–3 define "parallel circles" and illustrate some of their properties: all points lying on the surface of a sphere rotating uniformly about its axis—except those points on the axis itself (*i.e.*, at the poles)—will describe parallel circles that have the same poles as the sphere and are perpendicular to the axis. In propositions 2 and 3, it is argued that, under uniform motion, all points on the surface of the sphere trace similar arcs of the parallel circles on which they are carried in equal times.

Propositions 4–11 consider the motion of points on circles on the surface of the sphere with respect to the horizon. If the horizon is taken to be fixed *orthogonal* to the axis of the rotating sphere, none of the points on the surface of the surface will rise or set: those in the visible part will always be visible and those in the invisible part always invisible (prop. 4). If the axis of a rotating sphere lies on the fixed horizon, all points on its surface will rise and set and will be visible and invisible for equal times (prop. 5). If the horizon is taken to be *inclined* to the axis, several observations follow:

The horizon will lie tangent to two small circles on the surface of the sphere which are equal and parallel to one another: the circle towards the visible pole is always visible and the one towards the invisible pole is always invisible (prop. 6).[2] A small circle lying on the surface of the sphere perpendicular to the axis and intersecting the horizon will always rise and set at the same points and will be equally inclined to the horizon (prop. 7).The great circles lying on the sphere tangent to two small

1. For a discussion of definitions in the extant Greek manuscripts of *On the Moving Sphere*, see: Tannery, *Mémoires Scientifiques*, 54–59. See also: Árpád Szabó, *The Beginnings of Greek Mathematics* (Dordrecht; Boston: D. Reidel Publishing Company, 1978), 220–26.
2. For a discussion of this proposition see: Nathan Sidoli, "On the use of the term *diastēma* in ancient Greek constructions," *Historia Mathematica* 31, no. 1 (February 1, 2004): 2–10.

circles which are tangent to the horizon will coincide with the horizon at one moment as the sphere rotates (prop. 8). The points which rise closer to the visible pole will set later while those points which set closer to the visible pole will rise earlier (prop. 9).

The circle passing through the poles as the sphere rotates will be at two times perpendicular to the horizon (prop. 10).

If another great circle is tangent to parallel circles that are larger than those to which the horizon is tangent, the points on that second great circle rise and set on the part of the horizon that is bounded by those larger parallel circles to which the supposed great circle is tangent (prop. 11).

In the twelfth and final proposition, Autolycus considers the case in which a great circle bisects another circle and defines the conditions under which that latter circle would itself be a great circle.

The propositions in *On the Moving Sphere* generally follow a Euclidean form of proof. This form typically consist of the *πρότασις* (*protasis*, enunciation), followed by the *ἔκθεσις* (*ekthesis*, setting out), *διορισμός* (*diorismos*, definition of the goal), *κατασκευή* (*kataskeuē*, construction), *ἀπόδειξις* (*apodeixis*, proof), and ending with the *συμπέρασμα* (*sumperasma*, conclusion). The *protasis* is a statement in general terms of what is to be proved. The *ekthesis*-*diorismos* restate the *protasis* in terms of a lettered figure. The *diorismos* is an assertion usually beginning with 'I say that' and serves to make the proof easier to follow. The *sumperasma* delivers the inference to the original proposition from the construction and proof, repeating the protasis in general terms and ending with 'which was required to be proved.'[1]

This style of proof was, in some sense, appropriated into medieval Arabic and Persian mathematical works, even as various mathematicians adapted and modified this form according to their intentions in a particular text. This phenomenon is evident in the various translations and recensions of *al-Kura al-mutaḥarrika*. In version I, for example, in two separate places in the seventh proposition, the adapter adds *diorismoi* that are not found in the Greek text and which Ṭūsī later removes in his own recension. In other cases, we see that some adapters

1. This is the case, in general, for theorems but not for problems where the purpose of the demonstration is to confirm the construction and so Euclid typically ends with the phrase 'which was required to be done.' See: Heath, *Euclid*, 1:124-29.

include or remove auxiliary propositions needed in the course of the construction or proof. We even see adapters, of versions II and III in particular, attempting to state proposition 7 in more general terms by considering cases that we do not find in the extant Greek text nor in other Arabic versions.

That there were wide variations in the adaptations of the Euclidean style of proof into Arabic mathematical works was not lost upon these mathematicians themselves: Maghribī, in his recension of Euclid's *Elements*, known as *Taḥrīr al-Uṣūl*, criticizes Avicenna for having omitted the enunciations and many lemmata in his own recension of the *Elements*. Maghribī is similarly critical of the 11th century mathematician Abū al-Qāsim ʿAlī ibn Ismāʿīl al-Nisābūrī for doing the same and for including repetitive additions while omitting important details. Although Abū Jaʿfar al-Khāzin (d. *c*. 971) apparently included the enunciations, Maghribī was still critical of his alteration of the number of propositions and rearrangement of their ordering. For his part, Maghribī tells us that his goal was to produce an improved edition which supplied better proofs by clarifying ambiguities, removing redundancies, addressing deficiencies, and mentioning lemmata (*muqaddimāt*) required in the course of the demonstrations.[1]

To see this adaptive practice at play, we can examine the different ways various mathematicians rendered proposition 7 of *On the Moving Sphere* into their Arabic versions.[2] This is an especially fruitful case study since this proposition is among a series of propositions that consider the implications of supposing the horizon to be inclined with respect to the axis. Furthermore, the available Greek text itself omits some auxiliary constructions in the course of the seventh proposition which we find referenced in some Arabic recensions, but not in others. This offers us the opportunity to not only situate these versions relative to each other but also relative to the *Sphaerica* of Theodosius, of which at least two of our adapters, Ṭūsī and Maghribī, have their own recensions.

In the seventh proposition, Autolycus considers the case of a sphere

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1. A.I. Sabra, "Simplicius's Proof of Euclid's Parallels Postulate," *Journal of the Warburg and Courtauld Institutes* 32 (1969): 14–15.
2. We are in the process of preparing an edition and translation of the entire text for all available versions which will allow us to present a better analysis of the text and its tradition. The seventh proposition serves here as a case study.

with a horizon inclined to the axis and opens with the *protasis*, asserting that the small circles lying on the surface of the sphere perpendicular to the axis and intersecting the horizon will always rise and set at the same points and will be equally inclined to the horizon. Autolycus used the expression "*ὁ ὁρίζων ἐν τῇ σφαίρᾳ κύκλος τό τε φανερὸν τῆς σφαίρας καὶ τὸ ἀφανὲς καὶ τὸ ἀφανὲς*" (the circle in the sphere dividing the visible and the invisible [portions] of the sphere) to refer to the horizon circle. In version III, *ὁρίζω*—from a root meaning to divide, separate from, as a border or boundary—became *ḥadda* while in version II, it became *faṣala*. Al-Farghāī (fl. 9th century) in his *Jawāmiʿ* and Thābit ibn Qurra in his *Tashīl al-majisṭī* use the same verb as version II.[1] Versions V and I are here consistent with version III and, throughout those versions, the horizon circle is always referred to using this same formula. However, in the first appearance of the horizon circle in version IV, in proposition 4, Ṭūsī uses a similar expression, "*idhā kānat ʿalá kuratin dāʾiratun ʿaẓīmatun taḥiddu bayna ẓāhirihā wa-khafiyyihā*," but added "*wa-l-tusamma bi-al-ufuqi*" (let it be called 'the horizon'). Ṭūsī uses similar terminology in proposition 5, "*al-dāʾiratu al-ʿaẓīmatu ʿalá al-kurati al-fāṣilati bayna ẓāhirihā wa-khafiyyihā*", and again adds "*aʿnī al-ufuq*" (meaning, the horizon). In all propositions thereafter, including the seventh, Ṭūsī uses only the term *al-ufuq* to refer to the horizon.

Crucially, the *protasis* in versions II and III differs from that found in the extant Greek text and the other Arabic versions in that there is an attempt to cast the seventh proposition in more general terms, that is, to handle both the case in which the horizon is inclined toward the axis *and* the case in which it is passes through the poles of the sphere. Thus,

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1. Aḥmad ibn Muḥammad ibn Kathīr al-Farghānī, *Jawāmiʿ ʻilm al-nujūm wa-uṣūl al-ḥarakāt al-samāwīya*, ed. Jacob Golius, Amsterdam, 1669 (reprinted by Fuat Sezgin, Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften an der Johann Wolfgang Goethe Universität, 1986), 20:

«فنقول أنّ دائرة الأفق هي الدائرة الّتي تفصل بين ما يظهر من السماء فوق الأرض وبين ما يخفى منها تحت الأرض»

"then we say the horizon circle is the circle that divides what is visible of the sky above the earth from what is invisible below the Earth."

Thābit ibn Qurra, *Tashīl al-majisṭī*, ed. Régis Morelon, in *Thābit ibn Qurra Œuvres d'astronomie*, Paris: Société d'édition «Les Belles Lettres», 1987, 3:

«دائرة الأفق هي الدائرة الّتي تفصل بين ما يرى من السماء فوق الأرض وبين ما تحتها»

"the horizon circle is the circle that divides what is seen from the sky above the earth and what is below it."

the *protasis* of version II includes, "*kāna al-ufuqu māʾilan aw kāna ʿalá al-quṭbayni wa-idhā kāna al-ufuqu māʾilan ʿan al-miḥwar*" (the horizon being either inclined or [passing] through the poles; if the horizon is inclined with respect to the axis). In version III, this was stated as, "*fa-in kāna al-ufuqu māʾilan ʿalá al-miḥwar*" (then if the horizon is inclined toward the axis).

The Greek *ὁμοίως εἰσὶ κεκλιμένοι*, meaning 'to be similarly inclined,' used in the *protasis* and elsewhere in this proposition was rendered in version III as *maylun mutashābihun* and in version II as *mutasāwiya*, except in the closing section of the proof in which we find version III using *mutasāwiya* where II uses *mutashābihun*.[1] In versions I and IV, the term *mutashābih* was used consistently where *mutasāwī* was used in version V. Finally, whereas we consistently find the preposition *ʿalá* used for 'inclination toward' throughout versions I, III, IV, and V, in version II the preposition *ʿan* is preferred.

The *protasis* is followed by an *ekthesis* in which the horizon and the parallel circles are named, then the *diorismos* restates the *protasis* in terms of the lettered figure of the *ekthesis*. The *diorismos* in each of versions II and III differs from those of the Greek and other Arabic versions in a manner corresponding to the aforementioned differences in the enunciations, so the condition of the horizon being inclined with respect to the axis is removed in the *diorismos* in these two versions. Versions V and I mirror the *diorismos* found in the Greek text while Ṭūsī omits it altogether in version IV.

In versions II and III, like the Greek text, the first *diorismos* mentions only the first goal of the proposition, *i.e.*, the rising and setting of all parts of the perpendicular circles on identical points. However, the author of version V also asserts in this *diorismos* that the circles have equal inclination with respect to the horizon.

In the *ekthesis-diorismos*, the translation of *ὁρίζω* in all three versions follows the same style as that in the enunciation. The author of version II uses *fa-l-natruk* where the authors of other versions use *fa-l-yakun*, this is a stylistic feature which may help us in determining the authorship of this version. We also see here for the first time another distinctive stylistic feature, this time in version III, where the author

1. The corresponding phrase in the protasis is found in the Greek text as *ὁμοίως ἔσονται κεκλιμένοι* (they will be similarly inclined).

uses the term *marra*—meaning 'to pass through'—to describe the point at which the parallel circles rise and set with respect to the horizon. All other differences are just slight re-wordings with no significant change in meaning, except in version V where it has been specified that the perpendicular circles are parallel to each other.

Autolycus undertakes to prove this *diorismos*, first constructing the requisite figure and then supplying the necessary argument. The proof proceeds with a *reductio ad absurdum*: if we are to reject the assertion that the parallel circles rise and set on the same pair of points, then we must suppose that the rising and setting occurs at another pair of points. The proof supposes such an alternative point for the rising of the first circle and shows it to be absurd, so the rising point must be as claimed in the *diorismos*. In versions I, III, IV, and V, the term *ghayr* is used in referring to 'a point other than the specified point' while other expressions like *nuqṭatin ukhrá*, *nuqṭatin ukhrá siwá*, and *nuqṭatin ukhrá khalā* are used in version II. Autolycus and the Arabic adapters leave to the reader the task of applying the same procedure to prove the claim with respect to the rising of the second parallel circle. In the course of the construction and proof nothing is said of the setting point, but it is made clear at the end of the proof that the same procedure can once again be applied to prove that the setting point remains the same.

The task in the construction is to suppose the alternative rising point while the setting point remains the same—versions I, IV, and V omit an explicit statement of the latter, but all take it to be implicit. The first step is to suppose a certain great circle to be the horizon, inclined to the axis of the sphere. The next step is to posit a certain pole for the parallel circles which is also the pole of the sphere. Then, the reader is instructed to draw a great circle on the sphere that passes through the poles of the parallel circles and the *poles* of the horizon circle on the sphere. In the Greek text and in versions II, III, and V, the authors take for granted that it is possible to draw such a circle. However, in versions I and IV, the authors introduce a method for drawing that circle. Although neither author tells the reader why they have included this addition or from where it was drawn, it relates to two propositions of Theodosius' *Sphaerica*, I.21–22 (I.20-21g).[1] Proposition 7 does not require that this

1. As has been pointed out by Nathan Sidoli, the arrangement of the proposition in the edited version of the Arabic *Sphaerica* is slightly different from that in the extant Greek version which

auxiliary construction itself be proved as it sufficies to know that it is possible to draw such a circle and that the reader is probably aware of a method for doing so given in another text.

Sphaerica I.21 explains how to draw a great circle on a sphere that passes through any two given points on the sphere. *Sphaerica* I.22 gives a procedure for determining the poles of a known circle on the sphere. In our case, we need to suppose a given great circle on a sphere and a point on the sphere and then a procedure for drawing a great circle on the sphere that passes through the poles of the given great circle and the given point: we take the given point as the pole of a small circle which is drawn to be tangent to the given great circle (the existence of such a small circle was proved in proposition 6). Then, if we draw a great circle passing through the known point and the point of tangency,[1] this great circle *necessarily* passes through the poles of the given great circle. Using this construction, we can draw a great circle on the sphere that passes through a known point—the pole of the parallel circles on the sphere—and passes through the poles of a known great circle—the horizon circle. Since this great circle passes through the poles of the horizon circle, it will necessarily be perpendicular to the horizon circle.

In versions I and IV, this procedure is executed while assuming specifically *two* parallel equal circles both of which lie tangent to the horizon—the same parallel circles given previously in proposition 6. Since the poles of these two parallel circles is the same, the new great circle will also pass through the points of tangency between both parallel circle and the horizon. This feature is unique to versions I and IV, is not found in any other recension—Arabic or Greek—and results in a characteristically different diagram.

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resulted in different proposition numbering in books I and II. Through out this paper, the proposition numbering in the Greek version has been given in addition to the Arabic numbering in cases where the two differ. See: Nathan Sidoli, “Book review on Theodosius Sphaerica: Arabic and Medieval Latin Translations by Paul Kunitzsch and Richard Lorch,” *The Journal of the American Oriental Society* 133, no. 3, (July-September, 2013): 592-593.

1. In order to do this, we need first to determine the point of tangency itself. So the task would be: for a given great circle on a sphere and a given point on the sphere, which is supposed to be the pole of a small circle, find the point of tangency of that small circle and the given great circle. We could not find a proposition efecting this procedure in Theodosius’ *Sphaerica*, but proposition II.12 (II.14g) in Theodosius’ *Sphaerica* is the reverse of this procedure: if a small circle in sphere and some point on its circumference are given, draw through the point a great circle tangent to the small circle. See: Martin, 41-42.

The mathematization of astronomical problems in works like *On the Moving Sphere* is a point of great interest among historians of astronomy. Indeed, it is rare to find other mathematical texts invoking propositions from *On the Moving Sphere* in a geometrical context. However, although the sixth proposition has both astronomical and geometrical applications, its use here, in the course of proposition 7 in versions I and IV, reflects primarily geometric purposes.

The author of version V makes his own unique addition to the construction, saying that the newly drawn great circle will be perpendicular to the horizon and to the parallel circles ("*fa-hiya qāʾimatun ʿalá al-ufuqi wa-ʿalá tilka al-mutawāziyyāt*"). This observation is, at this point, omitted in versions II and III, as in the Greek text, although it will be brought up later in the proposition. Meanwhile, versions I and IV mention that the constructed great circle will be perpendicular to the horizon circle but leave implicit its similar relation to the parallel circles. This observation is related to proposition I.16 (I.15g) of Theodosius' *Sphaerica*: "If a great circle on a sphere cuts some circle on the sphere and passes through its two poles, it bisects it and at right angles."[1]

In versions II, III, and V, as in the Greek text, the final step in the construction is to draw four lines: (1) the diameter of the horizon circle, (2) a line from the pole of the sphere to the supposed rising point on the horizon, (3) another line from the pole of the sphere to the original rising point,[2] and (4) a line from the pole of the sphere to the point of intersection between the horizon and the previously drawn great circle. The second and third lines are asserted as equal in version V. All of these lines are drawn in versions I and IV, except the fourth line. Instead, the diameter of one of the small parallel circles is drawn from the point of tangency with the horizon.

Having completed the construction, the proposition proceeds with an *apodeixis* in which the goal is to show that the line extending from the pole of the sphere to the supposed rising point is shorter than the line extending from the pole to the rising point set out in the *diorismos*. This

1. Martin, 19; Kunitzsch and Lorch, 60.
إذا قطعت دائرة عظيمة في كرة دائرة ما من الدوائر الّتي في الكرة ومرّت بقطبيها فهي تقطعهما بنصفين وعلى زوايا قائمة.

2. Although this is not mentioned explicitly in version II at this point, the author later makes reference to this line.

starts with the observation that the great circle drawn earlier in the construction is perpendicular to and bisects the horizon circle, intersecting at the diameter.[1] In the Greek text, the verb *ἐφίστημι*—meaning set upon, stand upon—is used to characterize the relative position of the drawn great circle to the horizon. In version III, this verb was rendered as *ḥamala* while in version II it became *qāma*. In Thābit ibn Qurra's restoration of Euclid's *Data*, we find a similar usage of *qāma* corresponding to the Greek *ἐφίστημι*.[2] The wording of the *apodeixis* in these two versions is otherwise very similar to each other and both are similar in content to the Greek text.

The arc along the drawn great circle, extending from the intersection point at one end of the diameter to the other, is divided unevenly at the pole of the sphere. Since one of these divisions is less than half of the whole arc, the chord of this division is the shortest of all lines extending from the pole to the circumference of the sphere. This argument depends on yet another proposition, perhaps also from an earlier lost text on spherics, corresponding to *Sphaerica* III.1:

If on a circle some straight line is drawn cutting the circle into two unequal parts, and there is constructed on it a segment of a circle not greater than half of it, and it is set up on it at right angles, and the arc of the segment is constructed on the line is cut into two unequal parts, the line which subtends the smallest arc is the shortest of all the straight lines which are drawn from that point at which the arc is cut to the greatest arc of the first circle. Similarly, if the drawn line is a diameter of the circle, and the remaining characteristics, which hold for the segment that was not greater than half of the circle constructed on the line, are the same, the drawn line previously mentioned is the shortest of all the straight lines drawn from that same point reaching the circumference of the first circle and the greatest of them is the line which the greatest arc subtends.[3]

1. This follows from *Sphaerica* I.16 (I.15g).
2. See for example: Nathan Sidoli and Yoichi Isahaya, *Thābit Ibn Qurra's Restoration of Euclid's Data: Text, Translation, Commentary* (Cham, Switzerland: Springer, 2018), 127.
3. Martin, 78; Kunitzsch and Lorch, 208:
إذا خطّ في دائرة خطّ ما مستقيم يقسم الدائرة بقسمين غير متساويين وعمل عليه قطعة من دائرة ليست بأعظم من نصفها وكانت قائمة على الدائرة على زوايا قائمة وقسمت قوس القطعة الّتي عملت على الخطّ بقسمين غير متساويين فإنّ الخطّ الّذي يوتر القوس الصغرى يكون أقصر جميع الخطوط المستقيمة الّتي تخرج من تلك النقطة الّتي انقسمت القوس عليها

Since the supposed rising point along the horizon circle is closer to the shortest line from the pole to the circumference of the sphere than the rising point given in the *diorismos*, the line extending from the pole of the sphere to the former is shorter than the line extending from the pole to the latter. In the Greek text, the extension of the line from the pole to the circumference is described by the phrase *προσπιπτουσῶν εὐθειῶν* which is given as '*yaqaʿu min ... ʿalá ...*' in version II and as '*takhruju min ... wa-talqá bi-...*' in version III. The Greek term *ἐλαχίστη*—meaning least—is here and in the following sections rendered as *aṣghar* and *aqṣar* in versions II and III respectively. Both Arabic terms are used in version V. It should also be noted that the *apodeixis* found in version V is more concise than that of versions II, III, and the Greek text.

We are now in a position to complete the *reductio ad absurdum*. Both the supposed rising point and that given in the *diorismos* lie on the parallel circle. All lines drawn from the pole of a circle to any point on the circumference should be equal. But we have just seen that the line extending from the pole to the supposed rising point is shorter than that extending to the rising point given in the *diorismos*, so we have a contradiction. Consequently, the parallel circle does not rise on any point other than the one given in the *diorismos*. The same method of proof can be applied to the setting point and to the rising and setting of any other parallel circles. At the end of the *reductio ad absurdum* proof, the Greek expression *ὅπερ ἐστὶν ἄτοπον* is rendered in Arabic as '*hādhā khulfun lā yumkin*' in versions I and II and as '*wa-dhālika muḥāl*' in versions III and V. It is worth noting that Kindī used '*hādhā khulfun lā yumkin*' in a number of his works.[1]

As mentioned above, we find in versions II and III a treatment of the case in which the horizon passes through the poles of the sphere, which we do not find in the other Arabic versions or the Greek text. This entails a brief modification of the earlier construction and a

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إلى القوس العظمى من الدائرة الأولى. وكذلك أيضاً إن كان الخطّ المخرج قطر الدائرة وكانت سائر الأشياء الّتي كانت للقطعة الّتي ليست هي أعظم من نصف دائرة المعلومة على الخطّ على حالها فإنّ الخطّ المخرج الّذي تقدّم ذكره أقصر جميع الخطوط المستقيمة الّتي تخرج من تلك النقطة بعينها وتلقى الخطّ المحيط بالدائرة الأولى ويكون أعظمها الخطّ الّذي يوتر القوس العظمى.

1. See: ʿAbd al-Qādir Muḥammad ʿAlī, ed., *al-Rasāʾil al-falsafiyya li-l-Kindī* (Beirut, Dār al-kutub al-ʿilmiyya, 2018): 21, 28-30, 66.

corresponding *apodeixis* which proceeds in much the same way as the previous *reductio ad absurdum*.

The proposition now takes up the second part of the enunciation: arguing that the circles perpendicular to the axis are similarly inclined to the horizon. This section opens with a corresponding *ekthesis-diorismos* wherein the previous construction is used again except that a number of new lines are drawn connecting the intersections of the parallel circles with the drawn great circle and with the plane of the horizon. Now the task is to relate this configuration to a definition of the inclination of a plane with respect to another, similar to the seventh definition found at the beginning of the first book of the Arabic *Sphaerica*:

It is said that a plane is inclined to another plane if we mark on the common section of the two planes some point, and there is drawn from it to each one of the two planes a straight line at right angles to the common section, and so the two drawn lines contain an acute angle, and the inclination is the angle which those two straight lines contain.[١]

The eighth definition in the Arabic *Sphaerica* gives the conditions under which the inclination of two planes with respect to each other is similar to the inclination of two other planes with respect to each other.[٢] Interestingly, this definition was mentioned explicitly in version III: “And it is said that the surface is inclined relative to a surface with an inclination similar to the inclination of another surface [to the latter surface] if the lines that extend to their intersections at right angles encompass each of the two surfaces by equal angles.”[٣]

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1. Martin, 1; Kunitzsch and Lorch, 14.
يقال أنّ السطح مائل على سطح آخر إذا نعلم على الفصل المشترك للسطحين نقطة ما وأخرج منها في كلّ واحد من السطحين خطّ مستقيم قائم على الفصل الًمشترك على زوايا قائمة فأحاط الخطّان المخرجان بزاوية حادّة والميل هو الزاوية الّتي تحيط بها ذانك الخطان المستقيمان.

This definition does not exist in the Greek version of *Sphaerica*.

2. Martin, 1-2; Kunitzsch and Lorch, 14.
ويقال أنّ ميل السطح عن السطح مثل ميل سطحٍ آخر عن سطحٍ آخر إذا كانت الخطوط المستقيمة الّتي تخرج من الفصول المشتركة للسطوح على زوايا قائمة في كلّ واحدً من السطوح مًن نقطٍ باعيانها محيطة بزوايا متساوية والّتي تكن زواياها أصغر فهي أكثر ميلاً.

3. See section 10 of our edition in Appendix 3.1:
ويقال أنّ السطح يميل على سطحٍ ميلاً شبيهاً بميل سطحٍ آخر على سطحٍ آخر إذا كانت الخطوط الّتي تخرج إلى الفصول المشتركة لها على زوايا قائمة تحيطً في كلّ واحد من السطًحين بزوايا متساوية.

In the *apodeixis,* Autolycus tells us that the intersection line of each parallel circle with the horizon is perpendicular both to the diameter of the horizon and to the parallel circle's intersection with the drawn great circle. Then, the angle between the intersections of the parallel circles with the drawn great circle and the diameter of the horizon is the inclination of the parallel circles with respect to the horizon. Since the parallel circles intersect the drawn great circle, their intersections are parallel to each other and, thus, their inclinations to the horizon will be the same. The Greek term used to refer to the intersection of planes, *κοινὴ τομή*, is rendered in version I as '*al-taqāṭiʿāt al-mushtaraka*.' In all other Arabic versions, we find the term '*al-faṣl al-mushtarak*' and its plural variants.

In the second part of the proposition, the wordings of versions II and III are, once again, close to each other and similar to the Greek. Where there the Greek text has *ἅπτεται* (*ἁπτομένας*)—meaning 'touch'—version II has *tamāss* and version III has '*takhruju min*.' The Greek phrase *διὰ τὰ αὐτὰ δὴ*—meaning 'according to the same [reasons]'—is rendered as *wa-kadhālika* in version II and as '*li-hādhā al-ashyāʾ bi-aʿyānihā*' in version III. Finally, the Greek *ἐπίπεδον*—meaning 'plane'—is given as '*basīṭu saṭḥi*' in version II and simply as *saṭḥ* in version III.

Starting from the first *apodeixis* until the end of the proposition, versions I and IV are different than the other Arabic versions and yet similar to each other in ways which we will now highlight. To begin with, one feature unique to version I, in comparison to the other Arabic versions, is that the Greek term *ἐπεὶ*, meaning 'since,' is rendered as the Arabic '*min ajl*' whereas in versions II and III it is given as different forms of the Arabic *li-ann*. We find '*min ajl*' in the Arabic translations of other Greek texts prepared by Qusā ibn Lūqā (d. c. 912) as well.[1]

In versions I and IV, the *apodeixis* begins by showing that the small circle tangent to the horizon is parallel to the parallel circles supposed in the *ekthesis*, whose rising and setting points are the subject of this proposition. So, their respective intersections with the drawn great circle are also parallel and the angles between these parallel lines and the diameter of the horizon are equal. In these two versions, this point

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1. See: Paul Kunitzsch and Richard Lorch, "Theodosius, De diebus et noctibus," *Suhayl* 10 (2011): 16.

is made in the course of proving that the rising and setting points are consistently the same, whereas in versions II, III, and V, this point is not made until the second half of the proposition, on the inclination of the parallel circles.

In the course of the *apodeixis* in versions I and IV, we have a restatement of the *diorismos*, mentioning explicitly the two points of rising and setting. Then, the *apodeixis* resumes to supply the necessary argument by a *reductio ad absurdum*. It is possible to consider section 6 in the proposition outline below as part of the construction, not the *apodeixis*. This would make the proposition structure closer to what we expect from a Euclidean style of proof. However, the content of the section in question goes beyond what we normally expect of a construction since it relies on external lemmata and propositions to make statements about the construction, a feature usually reserved for the *apodeixis*. In any case, the boundaries between these various sections vary from one proposition to another and one author to another. In these two versions the *diorismos* follows, rather than precedes, the construction which is a common feature of Euclideian-style theorems.

As the *apodeixis* resumes, we suppose an alternative rising point and show that the line drawn from the pole of the sphere to the new point is shorter than that extending to the rising point given in the *diorismos*. This leads to a contradiction, since these two lines were supposed to be equal, and so the rising point must be as given in the *diorismos*. In mathematical terms, this is the same argument we find in versions II, III, and V—the major difference being that version IV is more concise, providing only the final result and its necessary conditions, omitting intervening steps in the proof.

In version I, the second portion of the proposition is introduced with another *diorismos*, omitted in version IV, followed by an *apodeixis*. The argument here differs noticeably from that given in versions II, III, and V in that it depends on different auxiliary constructions, uncited in both versions. The task here is to show that the two arcs along the horizon, from its intersection with the drawn great circle to each of the rising and setting points are equal—as are the arcs along the parallel circle—to its intersection with the drawn great circle. This is related to Theodosius' *Sphaerica* II.9: "If there are two circles on a sphere which cut one another, and a great circle is described passing through their poles, it bisects the segments which are cut off from the circles."[1]

In the course of comparing and collating the various Arabic versions of *al-Kura al-mutaḥarrika*, it is possible to identify two coherent pairs of versions: II and III; I and IV. We can say that one of the authors of versions II and III likely relied on the other version in preparing their own, but we cannot say definitively which preceded the other. The strong similarities between I and IV indicate that Ṭūsī used a text close to version I, if not this very text, in writing his recension. Although version V, Maghribī's recension, exhibits notable similarities to II and III, it is still sufficiently different from both that it is not possible to reasonably determine its relation to either. In fact, Maghribī introduces two propositions, between propositions 11 and 12 of Autolycus, which are not present in other versions. In the editions of the Arabic texts that follows, we have set in boldface the wording and phrasing that is common among each of the two pairs—II and III; I and IV—in order to highlight similarities and differences.

Several hypothesises can be put forth to explain the difference between version I and other early versions, that is, the extant Greek text and versions II and III. It may be that version I was a translation of a different Greek recension that has not reached us. It may also be that version I was a reworking of an earlier Arabic translation which itself was similar to versions II and III. Finally, it may be that version I was intended as a recension of the Greek text, not a direct translation. Ibn al-Nadīm credits Kindī with a correction of *al-Kura al-mutaḥarrika* and Kindī himself alludes to a work of his, which has not reached us, that may have well been that text. Can we draw a connection between Kindī and this version I? This would require further research but we may consider Kindī as a possible author for version I or perhaps a similar text.

On the other hand, we have the testimony in all manuscript witnesses of version IV that it is based on a correction by Thābit. So, shall we then attribute version I to Thābit given its similarity to IV? This is possible, but we have to consider contradictory evidence: MSS L and S (version

1. Martin, 34; Kunitzsch and Lorch, 106.
إذا كانت في كرة دائرتان تقطع إحداهما الأخرى ورسمت دائرة عظيمة تمرّ بأقطابهما فإنّها تقصم القطع الّتي فصلت من الدوائر بنصفين نصفين.

III) claim to be witnesses to a work of Thābit. Further still, the striking similarities between versions II and III bring to mind the cases of Greek mathematical works translated by Isḥāq and later corrected by Thābit. Indeed, we have copies of *al-Kura al-mutaḥarrika* that bear Isḥāq's name and others that bear Thābit's. So we should at least be open to the possibility that II-III represents a translation-correction of Isḥāq-Thābit, or some other pair of scholars. There is no evidence to directly contradict the attribution to Thābit in the copies of version III; however, the similarity of some of the terms in version II to those used by Thābit in his other works may lead us to attribute version II to him.

Versions I, II, and III are early texts whose authorship remains unknown. We have many such cases among Greek scientific texts produced during the translation movement, including the *Sphaerica* of Theodosius whose situation bears undeniable similarity to ours. According to Kunitzsch and Lorch, there are at least seven versions of *Kitāb al-ukar*, among which only the authorship of the later recensions (including versions by Ṭūsī and Maghribī) can be reliably determined.[1] One promising avenue to shed light on the activities and attributions of the early translators is to compare the style and distinctive features of various versions not just within one textual tradition but across related traditions. That is, to explore the possibility of identifying patterns of translation and recension across both the Arabic revisions of *On the Moving Sphere* and the contemporaneous revisions of the *Sphaerica* and other mathematical texts.

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1. Kunitzsch and Lorch, *Sphaerica,* 1–2.

Appendix 1.1: Translations of the Arabic Texts
Versions II, III, and V

Protasis

	Ἐὰν ὁ ὁρίζων ἐν τῇ σφαίρᾳ κύκλος τό τε φανερὸν τῆς σφαίρας καὶ τὸ ἀφανὲς λοξὸς ᾖ πρὸς τὸν ἄξονα, οἱ τῷ ἄξονι πρὸς ὀρθὰς ὄντες κύκλοι καὶ τέμνοντες τὸν ὁρίζοντα κατὰ τὰ αὐτὰ σημεῖα αἰεὶ τοῦ ὁρίζοντος τάς τε ἀνατολὰς καὶ τὰς δύσεις ποιοῦνται · ἔτι δὲ καὶ ὁμοίως ἔσονται κεκλιμένοι πρὸς τὸν ὁρίζοντα.
III	If there is, on a sphere, a great circle fixed on the sphere separating what is visible of the sphere and what is invisible and if there are on the sphere circles erected to the axis at right angles, cutting the horizon, then the rising and setting of these circles will always be on identical [pairs of] points on the horizon and if the horizon is inclined to the axis, their inclination to the horizon will be similar.
II	If a circle on a sphere divides what is visible of [the sphere] from what is invisible and if there are circles on right angles to the axis cutting the horizon, then their rising and setting will always be on identical [pairs of] points among those points that are on the horizon—the horizon being either inclined [with respect to the axis] or [passing] through the two poles. If the horizon is inclined to the axis, then the inclination of the parallel circles perpendicular to the horizon is equal.
V	The seventh proposition: [for] every great circle fixed on the surface of a sphere and inclined to the axis that separates between the visible [part of the] sphere and the invisible, and there are other parallel circles which are perpendicular to the axis and cut the horizon, their rising and setting are on identical [pairs of] points on the horizon and their inclination to the horizon is similar.

Ekthesis

	Ἔστω ἐν σφαίρᾳ κύκλος ὁρίζων τό τε φανερὸν τῆς σφαίρας καὶ τὸ ἀφανὲς ὁ ΑΒΔΓ λοξὸς ὢν πρὸς τὸν ἄξονα, οἱ δὲ τῷ ἄξονι πρὸς ὀρθὰς ὄντες κύκλοι ἔστωσαν οἱ ΑΒ ΓΔ.
III	Let the circle ABDG on a sphere separate what is visible of it and what is invisible and let the two circles that are erected to the axis at right angles be the two circles AB and GD.
II	We suppose on a sphere a circle dividing between its visible [part] and its invisible [part] and it is the circle ABGD. And the two circles that are perpendicular to the axis on right angles are AB and GD.
V	The example of this is: let the great circle ABGD be fixed on the surface of the sphere and inclined to the axis, separating between the visible [part of the] sphere and the invisible. Let the two circles AEB and GZD be parallel and perpendicular to the axis.

Diorismos

	Λέγω ὅτι οἱ ΑΒ ΓΔ κύκλοι κατὰ τὰ αὐτὰ σημεῖα αἰεὶ τοῦ ὁρίζοντος τάς τε ἀνατολὰς καὶ τὰς δύσεις ποιοῦνται, καὶ διὰ μὲν τῶν Δ Β σημείων τὰς ἀνατολὰς ποιοῦνται, διὰ δὲ τῶν Α Γ τὰς δύσεις.
III	I say that the rising of the parts of the two circles AB and GD and their setting will always be on identical [pairs of] points on the horizon: their rising will be as they pass through the two points D and B, and their setting will be as they pass through the two points A and G.
II	I say that the rising and setting of the two circles AB and GD will always be on identical [pairs of] points among those points on the horizon, the rising on points B and D and the setting on points A and G.
V	I say that they always rise on the points B and G and set on the points A and D. And their inclination to the horizon is equal.

Kataskeuē

	Μὴ γάρ, ἀλλ’ εἰ δυνατόν, ποιείσθω ὁ ΑΒ κύκλος δι’ ἄλλου τινὸς σημείου τὴν ἀνατολὴν τοῦ Ε, διὰ δὲ τοῦ Α τὴν δύσιν, καὶ ἔστω ὁ πόλος τῶν παραλλήλων κύκλων τὸ Ζ σημεῖον, καὶ διὰ τοῦ Ζ καὶ τῶν τοῦ ΑΒΔΓ κύκλου πόλων μέγιστος κύκλος γεγράφθω ὁ ΗΖΘ, καὶ ἐπεζεύχθωσαν αἱ ΗΘ ΗΖ ΖΕ ΖΒ.
III	If this were not the case, then another is possible: let the rising of some of the circle AB be as it passes through the point E and its setting as it passes through the point A. Let one of the poles of the parallel circles be the point Z and we draw a great circle that passes through the point Z and the poles of the circle ABDG, which is the circle HZT. We connect the straight lines HT, HZ, ZE, and ZB.
II	If it is not like that, then let the rising of the circle AB be possibly on a different point, E, and its setting on the point A. Let the pole of the parallel circles be Z. We draw on the point Z and the two poles of the circle ABDG a great circle, that is the circle HZT. We connect the lines HZ, HT, and ZE.
V	Its proof is that: if the circle AB does not rise on B, then it rises on M. Let the pole of the sphere be L. Draw a great circle passing through the pole [of the sphere] and the pole of the circle ABGD, that is the circle ILK. It is perpendicular to the horizon and to all parallel circles. We also connect the lines LI, LM, and LB.

Apodeixis

	Ἐπεὶ ἐν σφαίρᾳ μέγιστος κύκλος ὁ ΗΖΘ κύκλον τινὰ τῶν ἐν τῇ σφαίρᾳ τὸν ΑΒΔΓ διὰ τῶν πόλων τέμνει, δίχα τε αὐτὸν τεμεῖ καὶ πρὸς ὀρθάς. Διάμετρος ἄρα ἐστὶν ἡ ΗΘ τοῦ ΑΒΔΓ κύκλου καὶ ὁ ΗΖΘ κύκλος ὀρθός ἐστι πρὸς τὸν ΑΒΔΓ κύκλον. Κύκλου δή τινος τοῦ ΑΒΔΓ ἐπὶ διαμέτρου τῆς ΗΘ τμῆμα κύκλου ὀρθὸν ἐφέστηκεν τὸ ΗΖΘ, καὶ ἡ τοῦ ἐφεστῶτος τμήματος τοῦ ΗΖΘ περιφέρεια εἰς ἄνισα τέτμηται κατὰ τὸ Ζ σημεῖον, καὶ ἔστιν ἐλάσσων ἡ ΖΗ περιφέρεια ἢ ἡμίσεια · ἡ ΖΗ ἄρα εὐθεῖα ἐλαχίστη ἐστὶν πασῶν τῶν ἀπὸ τοῦ Ζ σημείου πρὸς τὸν ΑΒΔΓ κύκλον προσπιπτουσῶν εὐθειῶν. Καὶ ἡ ἔγγιον ἄρα τῆς ΖΗ τῆς ἀπώτερον ἐλάσσων ἐστίν.

III	Since the great circle HZT is on a sphere and cuts the circle ABDG, which is one of the circles that is on the sphere, and passes through its poles, it cuts it in two halves at right angles. So, the line HT is the diameter of the circle ABDG and the circle HZT is erected to the circle ABDG at right angles. The section HZT has been set upon (*taḥammal*) the diameter HT of the circle ABDG and is erected to it at right angles. The arc HZT is divided in unequal divisions at the point Z and the arc ZH is smaller than half of [HZT] so the line ZH is the shortest of all lines extending from the point Z and meeting the circumference line of the circle ABDG. Thus, the line that is closer to the line ZH is shorter than the line that is further from it.
II	Since in the sphere there is a great circle—which is the circle HZT—that cuts from the circles within it—which is the circle ABDG—[passing] through its poles, then it cuts it in two halves on right angles. So the line HT is the diameter of the circle ABGD and the circle HZT is erected to the circle ABDG on right angles. Since the section HZT has been erected to the diameter of the circle ABDG—which is the line HT—at right angles and [since] the arc of the erected section—which is the arc HZT—has been divided in two unequal sections on the point Z and [since] the arc ZH is less than the half, the line HZ is thus smaller than all lines lying from the point Z to the circle ABDG. And the line that is closer than ZH is smaller than the line that is further from it.
V	The line LM equals the line LB since the circle AB rises at M and because of their extension from the pole to the circumference. And also since the circle ILK is perpendicular to the circle ABGD and the arc IL is less than half the arc IK, so the line IL is the shortest line that extends from the point L to the circumference of the circle ABGD and what is closer to it is smaller than what is further.

Apodeixis (conclusion)

	Ἐλάσσων ἄρα ἐστὶν ἡ ΖΕ τῆς ΖΒ · ἀλλὰ καὶ ἴση, ὅπερ ἐστὶν ἄτοπον. Οὐκ ἄρα ὁ ΑΒ κύκλος δι’ ἄλλου τινὸς σημείου ἢ διὰ τοῦ Β τὴν ἀνατολὴν ποιήσεται, διὰ δὲ τοῦ Α τὴν δύσιν. Ὁμοίως δὴ δείξομεν ὅτι καὶ ὁ ΓΔ κύκλος διὰ μὲν τοῦ Δ τὴν ἀνατολὴν ποιήσεται, διὰ δὲ τοῦ Γ τὴν δύσιν. Ὥστε οἱ ΑΒ ΓΔ κύκλοι αἰεὶ κατὰ τὰ αὐτὰ σημεῖα τοῦ ὁρίζοντος τάς τε ἀνατολὰς καὶ τὰς δύσεις ποιοῦνται.
III	So, the line ZE is shorter than the line ZB but it was [supposed to be] equal to it, which is impossible. So, the rising of any part of the circle AB is not at its passing through any point other than the point B. Nor is its setting at its passing through any point other than the point A. We [can] show, likewise, that the rising of all parts of the circle GD is at its passing through the point D and its seting is at its passing through point G. So, the rising of all parts of circles AB and GD and its setting a always at identical points on the horizon circle.
II	So, the line ZH is smaller than the line ZB but it was [supposed to be] equal to it which is an impossible contradiction. So, the circle AB does not rise on a point other than the point B. And it does not set on a point other than A. We [can] show, likewise, that the circle GD rises on the point D and sets on the point G. It necessarily follows from this that the two circles AB and GD rise and set on identical points among the points on the horizon.
V	So, the line LM is shorter than the line LB but they were [supposed to be] equal which is impossible. So, the circle AB does not rise except on the point B. This is the argument for the circle GD, that it does not rise on other than the point G. They always rise on two identical points on the horizon and set on two identical points.

Apodeixis

III	It is clear, then, that if we place the pole on the circle AGDB and we make it the point H, then the rising of the circle AB will be at its passing through the point B and its setting at its passing through the point A. This is because if it rises when it passes through the point E, then the line HE will also be equal to the line HB since they are extended from the pole to the circumference line of the circle, and this is impossible.
II	It is also clear that if we made the pole on the circle ABDG then the circle AB in the [previous] illustration would rise on the point B and set on the point A, since when it rises on the point E, the line EH would then be smaller than the line BH but the line EH is equal to HB since both of them [extend] from the pole, which is an impossible contradiction.

Ekthesis, diorismos, kataskeuē

	Λέγω δὴ ὅτι καὶ ὁμοίως εἰσὶ κεκλιμένοι πρὸς τὸν ΑΒΔΓ ὁρίζοντα. Ἐπεζεύχθωσαν γὰρ αἱ ΑΒ ΓΔ ΚΜ ΛΝ.
III	Let the circle ABDG be inclined to the axis. I say that the inclination of the two circles AB and GD to the horizon ABDG is similarly inclined. So, we connect the straight lines AB, GD, KM, and LN.
II	Again, let the circle ABDG be inclined to the axis. I say that the inclination of the two circles AB and DG on the horizon is equal. The proof of this is we connect the lines AB, GD, KX, and LM.
V	I say also that their inclination to the horizon is equal. The proof of this is that we extend the intersections of all circles, which are the lines AB, GD, IK, EH, and ZT.

Apodeixis

	Ἐπεὶ ὁ ΗΖΘ κύκλος τοὺς ΑΒ ΓΔ ΑΒΔΓ κύκλους διὰ τῶν πόλων τέμνει, καὶ πρὸς ὀρθὰς αὐτοὺς τεμεῖ. Ὁ ΓΖΘ ἄρα κύκλος ὀρθός ἐστι πρὸς ἕκαστον τῶν ΑΒ ΓΔ ΑΒΔΓ κύκλων. Ὥστε καὶ ἑκάτερος τῶν ΑΒ ΑΒΔΓ κύκλων ὀρθός ἐστιν πρὸς τὸν ΗΖΘ. Καὶ ἡ κοινὴ ἄρα τομὴ ἡ τῶν ΑΒ ΓΔΒΑ ἡ ΑΒ ὀρθή ἐστι πρὸς τὸν ΗΖΘ κύκλον. Καὶ πρὸς πάσας ἄρα τὰς ἁπτομένας αὐτῆς ἐν τῷ ΗΖΚΘ ἐπιπέδῳ ὀρθή ἐστιν ἡ ΑΒ. Ἅπτεται δὲ τῆς ΑΒ ἑκατέρα τῶν ΗΘ ΚΜ οὖσα ἐν τῷ τοῦ ΗΖΘ κύκλου ἐπιπέδῳ. Ἡ ΑΒ ἄρα πρὸς ἑκατέραν τῶν ΗΘ ΚΜ ὀρθή ἐστιν · ὥστε ἡ ὑπὸ τῶν ΚΜΘ γωνία ἡ κλίσις ἐστὶν ἐν ᾗ κέκλιται ὁ ΑΒ κύκλος πρὸς τὸν ΑΒΔΓ κύκλον. Διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ τῶν ΛΝΘ γωνία ἐστὶν ἡ κλίσις ἐν ᾗ κέκλιται ὁ ΓΔ κύκλος πρὸς τὸν ΑΒΔΓ.
III	Since the circle HZT cuts the two circles AGDB and AKB and passes through their poles and cuts them at right angles, then the circle HZT is erect upon each of the two circles AGDB and AKB at right angles. So, each one of the two circles AGDB and AKB are also erected on the circle HZT at right angles. Then, their intersection, which is the line AB, is erected on the circle HZT at right angles. So, [AB] is erected at right angles to all lines that extend from it and are on the surface HZT. And each of the two lines HT and KM, which are on the surface of the circle HZT, are extended from a point on the line AB. So, the line AB is erected upon each one of the two lines HT and KM at right angles. So, the angle KMT is the inclination by which the circle AB is inclined to the circle ABDG. And according to these same things, the angle LNT is the inclination by which the circle GD is inclined to the circle ABDG.
II	Since the circle HZT cuts the circles AGDB, AB, and GD on their poles, it bisects them at right angles. The circle HZT is perpendicular to each of the circles AGDB, GD, and AB at right angles. It is necessary then that each of the circles AGDB, AB, and GD be perpendicular to the circle HZT on right angles. Their intersections, which are the two lines AB and GD, are perpendicular to the circle HZT on right angles. They are also perpendicular to all lines on the surface HZT which touches them [i.e., the intersection lines], at right angles. And the line

	AB touches each of HT and LM, which are on the surface of the circle HZT. So, the line AB is perpendicular to each of the lines HT and LM at right angles. So, the angle LMT becomes the inclination of the circle AB to the circle ABDG. Likewise, the angle XKT is the inclination of the circle GD to the circle ABDG.
V	Since the circle AEB is parallel to the circle GZD and the surfaces of the two circles EZK and ABGD intersect them, the line EH is parallel to the line ZT and the angle EHT is like the angle ZTK. And also, since the circle IEK is perpendicular to all circles, they are also perpendicular to it. So, the two lines BH and GT are perpendicular to the surface of the circle IEZ. Each of the angles BHE, BHT, GTZ, and GTK are right angles. The angle EHT is the inclination of the circle AEB to the horizon, I mean the circle ABGD. And the angle ZTK is the inclination of the circle GZD to the horizon.

Apodeixis

	Καὶ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΑΒ ΓΔ ὑπό τινος ἐπιπέδου τοῦ ΗΖΘ τέμνεται, αἱ κοιναὶ ἄρα αὐτῶν τομαὶ αἱ ΚΜ ΛΝ εὐθεῖαι παράλληλοί εἰσιν · ὥστε ἴση ἐστὶν ἡ ὑπὸ τῶν ΚΜΝ γωνία τῇ ὑπὸ τῶν ΛΝΘ γωνίᾳ. Καὶ ἔστιν ἡ μὲν ὑπὸ τῶν ΚΜΘ γωνία ἡ κλίσις ἣν κέκλιται ὁ ΑΒ κύκλος πρὸς τὸν ΑΒΔΓ κύκλον, ἡ δὲ ὑπὸ τῶν ΛΝΘ γωνία ἡ κλίσις ἣν κέκλιται ὁ ΓΔ κύκλος πρὸς τὸν ΑΒΔΓ κύκλον. Οἱ ΑΒ ΓΔ ἄρα κύκλοι ὁμοίως εἰσὶ κεκλιμένοι πρὸς τὸν ΑΒΔΓ κύκλον.
III	Since the two parallel surfaces AB and GD are cut by a surface, which is the surface HZT, and [since] their intersections, which are the lines KM and LN, are parallel, so the angle KMT is equal to the angle LNT. And it is said that the surface is inclined relative to a surface with an inclination similar to the inclination of another surface to another surface if the lines that extend to their intersections at right angles encompass each of the two surfaces by equal angles. The angle KMN is equal to the angle LNT. The angle KMT is the inclination by which the circle AB is inclined to the circle ABDG. The angle LBZT is the inclination by which the circle GD is inclined to the circle

	ABDG. So, the inclination of the two circles AB and GD to the circle ABDG is equal and this is what we desired to show.
II	Since the surfaces of two parallel planes—they are AB and GD—are cut by the surface of a plane—that is the circle HZT—their intersections—which are KX and LM—are parallel, so the angle LMT becomes equal to the angle XKT. And the angle LMT is the inclination of the circle AB to the circle ADG and the angle XKT is the inclination of the circle GD to the circle ABDG. So, the two circles AB and GD are thus similarly inclined to the circle ABDG and this is what we desired to show.
V	Since angles are like angles, inclinations are like inclination, and such is the statement regarding all parallel circles and that is what was desired.

Appendix 1.2: Translations of the Arabic Texts
Versions I and IV

Protasis

I	If there is, on a sphere, a fixed great circle which separates between the visible and invisible [parts] of the sphere and which is inclined to the axis, and if other circles perpendicular to the axis intersect the horizon, the rising and setting of these circles are on the same [pair of] points on the horizon. Moreover, the inclination of these circles to the horizon is similar.
IV	If the horizon circle is inclined to the axis and is intersected by circles to which the axis is perpendicular, the rising and setting on the horizon of the points on these circles are on the same [pairs of] points. Moreover, the inclination of these circles to the horizon is similar.

Ekthesis

I	The example of this is that we imagine a great circle on a sphere, fixed upon it, separating between the visible [part] of the sphere and its invisible [part], that is circle ABGD, taken to be inclined to the axis. Let the two circles BED and ZHT be perpendicular to the axis, intersecting the horizon.
IV	Let the horizon be ABGD, which is inclined to the axis, and [let] the two circles BED and ZHT, to which the axis is perpendicular, intersect the horizon.

Diorismos

I	I say, the setting and rising of the two circles BE and ZH on the horizon are on a single identical point [pair] and their inclination to the horizon is similar.

Kataskeuē

I	The proof for this is: since the circle ABGD is inclined to the axis, it is tangent to two equal parallel circles whose poles are the pole of the sphere. Let these two circles be AKL and GM. Let the apparent pole of the sphere be the point S which is also the pole of the circle AKL. We draw through the two points A and S a great circle which necessarily passes through the poles of ABGD, to which it is perpendicular. Since the great circle passes through the poles of the circle GM it thus passes through the point G. Let the great circle be the circle ASKEHG.
IV	Let the horizon be tangent to the two circles AKL and GM. Let the apparent pole be S. Draw a great circle through A and S, it passes through the pole of the circle ABGD and is erected to it perpendicularly. Since [the great circle] passes through the pole of the circle GM, it [also] passes through the point G. Let the great circle be the circle ASKEHGM.

Kataskeuē

I	We extend the intersections of the surfaces, which are, the line BFD, AFNG, ZNT, EF, NH, AK.
IV	Let the intersections of the surfaces be BFD, ZNT, AG, KA, FE, NH.

Apodeixis

I	Since the poles of the circle AKL are the poles of the sphere, it is perpendicular to the axis. And the two circles BED and ZHT are perpendicular to the axis, so the circles AK, ZHT, and BED are parallel and are cut by a single surface, that is, the surface of the circle ASKEHG. Thus, their intersections are parallel. So, the lines AK, FE, and NH are parallel. So, the angle FAK is equal to the angle NFE; and the angle FAK is acute so the angle NFE is [also] acute.
IV	Because the circles AK, BD, and ZT, are parallel, the intersections AK, FE, NH are [also] parallel. Thus, the angle FAK is equal to the angle NFE; and the angle FAK is acute so the angle NFE is [also] acute.

Diorismos

I	I say that, if the sphere rotates, the circle BED cannot meet the circle ABGD except at the two points B and D.
IV	We say that the circle BED in its rotation does not meet the circle ABGD except at the two points B and D.

Apodeixis

I	If it were possible, then [the circle BED] would meet [the circle ABGD] at another point like the point Q. We extend from the pole S the lines SQ and SD so the lines SD and SQ are equal. Since the point S is the pole, the arc SA is equal to the arc SK. So, the arc GHEKS is greater than the arc SA and, since an arc of a circle, that is GA, is erected to the circle ABGD on its diameter and [since] the arc SA, which is less than half of it, is cut from it, thus the line which is its chord is the shortest line that is extended from this point [S] and meets the circle ABGD. And what is closer to it is shorter than what is further from it.
IV	Otherwise, [the circle BED] would meet [the circle ABGD] at Q. We connect SQ and SD, which are equal. Since the arc AEG, on the diameter AG, is perpendicular to the circle ABGD, and AS is smaller than half of it, the chord AS is the shortest line extending from S to the circumference of the circle ABGD.

Apodeixis

I	Thus, the line SQ is shorter than the line SD, although it was [supposed to be] equal to it, which is an impossible contradiction. Thus, the circle BED does not meet the circle ABGD at any point other than the points B and D. Thus, its rising and setting on the horizon is always on these two points. Likewise for the circle ZHT.
IV	SQ is shorter than SD but they were [supposed to be] equal which is a contradiction. Thus, the rising and setting of the points on the circle BED cannot be on other than the points B and D.

Diorismos

I	I say that the circles BED and ZH are similarly inclined to the circle ABGD.

Apodeixis

I	The proof of this is that, since the two circles ABGD and BED are intersecting and a great circle was drawn on their poles, that is the circle ASKE, then it cut their arcs at their midpoint. So, the arc DA is equal to the arc AB. And the arc BE is equal to the arc DE. Since the arc AB is equal to the arc AD and the diameter AFNG is extended, the line BF is equal to the line FD and the angle DFE is a right angle. And also, since the line BF is equal to the line FD and the arc BE is equal to the arc ED and that the line FE is extended, the angle DFE is a right angle. Since the surfaces ABGD and BED intersect, and on the surface ABGD the line FNG and on the surface BED the line FE, are erected on the intersection of these two surfaces and these lines encompass an acute angle, which is the angle NFE, then the angle NFE is the inclination of the surface BED on the surface ABGD. And the angle NFE is equal to the angle GNH. Thus, the inclination of the surface BE and [that of] HZ is a similar inclination, and this is what we desired to show.
IV	And also, since the circle AEG passes through the poles of the two circles ABGD and BED, which intersect each other, it bisects their [arc] sections. So, AB and AD are equal. And likewise, BE and ED. The diameter AG bisects BD at F and is perpendicular to it. Since the arcs BE and ED are equal, as are BF and FD, EF is also perpendicular to BD. Since FE and FG are perpendicular to the intersection BD and they are on the surfaces of the two circles ABGD and BED, the angle EFG is the inclination of the surface of the circle BED on the surface of the circle ABGD. Likewise, the angle GNH is the inclination of the surface of the circle ZHT on the surface of the circle ABGD. Because of the equivalence of the angles EFG and HNG, the inclinations are similar, and this is what we desired.

Appendix 2: Description of the Manuscripts

Version I

- **A (آ):** Turkey, Istanbul, Topkapı Saray Müzesi Kütüphanesi, Ahmet III, 3464. This codex contains the Arabic translation of most of the middle books. It has been used for the edition of several of these works.[1] The texts have been copied by different scribes. Autolycus' complete text with diagrams, which is a copy of our version I, is found on ff. 54b–58b. It does not have a copy date, but some of the other treatises copied by the same scribe bear the date 625/1228.
- **B (ب):** Iran, Tehran, Dānishgāh-i Tihrān, Kitābkhāna-yi Markazī, 1063. This codex was copied fairly late. In addition to a copy of version I, it contains some of Ṭūsī's recensions, including his recension of Autolycus' *On the Moving Sphere*. The folios are numbered on the upper left edge, which are indicated in the following list of items in the codex.
  Taḥrīr kitāb al-mafrūḍāt li-Thābit ibn Qurra, Naṣīr al-Dīn Ṭūsī, ff. 1b–7a;
  Fī tarbīʿ al-dāʾira, Ibn al-Haytham (d. *c*. 1040), ff. 7a–9b;
  Taḥrīr al-kura al-mutaḥarrika li-Ūṭūlūqus, Ṭūsī, ff. 10a–13b;
  Taḥrīr kitāb Ūṭūlūqus fī al-ṭulūʿ wa-al-ghurūb, Ṭūsī, 14a–20b;
  Same text, one folio is removed (item 7), ff. 21a–26a;
  [blank added paper];
  Same text, one folio in which the text follows the above item 4, ff. 20a–b(m);
  al-Kura al-mutaḥarrika li-Ūṭūlūqus, ff. 27a–28a;

1. For the description of the MS and its contents see: Roshdi Rashed and Athanase Papadopoulos, *Menelaus' Spherics: Early Translation and al-Māhānī/al-Harawī's Version* (Berlin; Boston: De Gruyter, 2017), 493-496; Richard Lorch, *Thābit ibn Qurra on the Sector-Figure and Related Texts*, ed. Fuat Sezgin, Islamic Mathematics and Astronomy 108 (Frankfurt am Main: Institute for the History of Arabic-Islamic Science at the Johann Wolfgang Goethe University, 2001), 21-23; Elaheh Kheirandish, *The Arabic Version of Euclid's Optics: Edited and Translated with Historical Introduction and Commentary* (New York, NY: Springer, 1999), xxvi; Nathan Sidoli and Yoichi Isahaya, *Thābit Ibn Qurra's Restoration of Euclid's Data: Text, Translation, Commentary* (Cham, Switzerland: Springer, 2018), 27-28; Nathan Sidoli and Takanori Kusuba, "Al-Harawī's Version of Menelaus' *Spherics*," *Suhayl* 13 (2014): 160-161; Paul Kunitzsch and Richard Lorch, *Theodosius Sphaerica: Arabic and Medieval Latin Translations* (Stuttgart: Steiner, 2010), 3; Paul Kunitzsch and Richard Lorch, "Theodosius, De diebus et noctibus," *Suhayl* 10 (2011): 13.

Taḥrīr kitāb Ibsiqlāʾus fī al-maṭāliʿ, Ṭūsī, ff. 28a–30a;
Taḥrīr kitāb al-maʾkhūdhāt li-Arashmīdis, Ṭūsī, ff. 30a–37b.

The copy of version I, item 8, is incomplete and starts at the middle of the 10th proposition and continues up to the end of the text. Hence this witness has not been used for the current edition of the 7th proposition.

- **K (ك):** Private Collection (formerly in possession of Paul Kraus).[1] This well-known codex, copied in 7th/13th century, contains most of the middle books and some other related works. Autolycus' text exists on ff. 65b–70b with diagrams.
- **X (خ):** United Kingdom, Oxford, Bodleian Library, Huntington, 237.[2] The codex contains the following texts:

  Sharḥ muṣādarat kitāb Uqlīdis fī uṣūl handasa, Ibn al-Haytham, ff. 1b–72a;
  Kitāb al-kuraẗ wa-al-usṭuwānaẗ li-Arašmīdus, ff. 73a-76a;
  al-Kurat al-mutaḥarrika li-Ūṭūlūqus, ff. 76a–82a;
  Iṣlāḥ Kitāb al-Makhrūṭāt, Abū Jaʿfar al-Khāzan, ff. 82a–104b;
  Bayān maʿānī kayfiyyat al-raṣd al-muḥaqqaq, ff. 104b–123a.[3]

The whole codex is copied by one person. The copy date, 8 Rajab 987/30-31 August 1579, is mentioned at the end of the last witness, on f. 123a. The Autolycus text, on ff. 76a–82a, begins by stating that this is the forth work of the middle books. The space for the diagrams in Autolycus' text is left blank.

Version II

- **F (ف):** Turkey, Istanbul, Köprülü Kütüphanesi, Fazıl Ahmed Paşa, 932. The items in the codex have been written by one hand probably in the late 7th/13th or early 8th/14th century. Some of the folios of the codex are omitted. Three texts are found in the codex:

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1. Regarding this MS, see the references in the note above for MS A. See also Elaheh Kheirandish, "A Report on Iran's 'Jewel' Codices of Tusi's Kutub al-Mutawasitat," in *Naṣīr al-Dīn Ṭūsī: philosophe et savant du XIIIe siècle. Actes du colloque tenu à l'université de Téhéran (6-9 mars 1997)*, ed. N. Pourjavady and Živa Vesel (Téhéran: Institut français de recherche en Iran, IFRI; Presses universitaires d'Iran, 2000), 131–45.
2. Heinrich Suter, *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig: B.G. Teubner, 1900), 40.
3. Another copy of this text exists in Qom, Marʿashī, MS 13757.

Taḥrīr al-majisṭī, Ṭūsī, ff. 1b–65b;
al-Kurat al-mutaḥarrika li-Ūṭūlūqus, ff. 66a–66b;
Fī shakl al-zuhra fī al-faṣl al-thānī min al-maqālat al-ʿāshira min al-majisṭī, Ṭūsī, ff. 67a.

It is not clear how many folios of the codex are removed; however, it seems that Autolycus' text was on three folios, only one of which remains in the codex. The present text starts at the middle of proposition 6 and ends at the middle of proposition 10. This is the only known copy of Autolycus' Arabic version II. The last item in the codex is one of Ṭūsī's short treatises which we find usually at the end of the copies of *Taḥrīr al-majisṭī*. This copy of Autolycus' text contains some cross-references to the relevant propositions in Theodosius' *Sphaerica*.

Version III

- **S (س):** Turkey, Istanbul, Süleymaniye Kütüphanesi, Ayasofya, 2671. This codex contains six works, all mathematical or astronomical in subject matter, copied by the same hand. The date *Ṣafar* 621/March 1224 is found at the end of the first witness, on ff. 75a.
  ʿAmal al-ṣafīḥa al-Zarqāliyya, Abū Isḥāq Ibrāhīm ibn Yaḥyá al-Naqqāsh al-Zarqālī (d. 1100), ff. 1b–76a;
  Kitāb Baṭlamyūs fī tasṭīḥ basīṭ al-kura, ff. 76b–97a;[1]
  ʿAmal bi-al-uṣṭurlāb, Abū al-Ḥasan Kūshyār ibn Labbān Bāshahrī al-Jīlī (*fl*. second half of 10th century), ff. 98b–122a;
  al-Kura al-mutaḥarrika, ff. 122b–132a;
  Risāla fī ʿamal al-uṣṭurlāb, ff. 133b–150a;[2]
  Risāla fī ʿamal al-rubʿ, ff. 150b–151b.

Item 4 in this codex represents our version III of Autolycus' work. Although the copyist left empty places for diagrams, none of them are drawn in the witness. There are a number of issues in the Arabic text in

1. One of the works in this codex was published in a facsimile edition in Christopher Anagnostakis, "The Arabic Version of Ptolemy's *Planisphaerium*" (Ph.D., Yale University, 1984) and later as a critical edition in Nathan Sidoli and J. L. Berggren, "The Arabic Version of Ptolemy's *Planisphere* or *Flattening the Surface of the Sphere*: Text, Translation, Commentary," *SCIAMVS: Sources and Commentaries in Exact Sciences* 8 (2007): 37–139.
2. David King argues that this treatise belongs to al-Zarqālī, see David A. King, "On the Early History of the Universal Astrolabe and the Origin of the Term 'Shakkāzīya' in Medieval Scientific Arabic," *Journal for the History of Arabic Science* 3, no. 2 (1979): 244–57.

general and especially in the lettering of the geometric objects (undrawn) which exhibits a lot of mistakes, likely due to scribal error.

- **L (ل):** United Kingdom, London, Institute of Ismaili Studies, Hamdani Collection, 1647. The folios of the codex are misordered and two folios of the Autolycus text, which is the second witness in the codex, are now found among the folios of the first witness. The codex contains three texts:
  Kitāb al-ukar li-Thāwdhūsiyūs, ff. 1b, 3a-12b, 14a-38b (the same version edited by Martin and Kunitzsch&Lorch);
  al-Kura al-mutaḥarrika, ff. 39b–40b, 2a-b, 13a-b, 41a-42b;
  Kitāb Uqlīdis fī ikhtilāf al-manāẓir, ff. 43b–52b (edited by Kheirandish).[1]

The diagrams of Autolycus' text are drawn in this copy. There are problems in the text in the lettering of the geometric objects, some of which are similar to the problems in MS S. Overall, this copy is more reliable.

Version IV

- **G (ج):** Iran, Tehran, Sipahsālār (Muṭaharrī), 4727.[2] This codex is a collection of 26 works, most of them by Ṭūsī, including his recensions of the middle books. The recension of Autolycus' text, which is on ff. ff. 67b-69b, has been copied in 17 Shaʿbān 671/9-10 March 1273 by Muḥammad ibn Aḥmad al-Zarkashī.
- **H (ح):** Turkey, Istanbul, Hacı Selim Ağa Kütüphanesi, Hacı Selim Ağa, 743.[3] It seems this codex was originally two separate codices that were later bound together. The first part, 135 folios, contains a collection of Ṭūsī's recensions of middle books and some other related works. This was copied in Rabīʿ I 1138/Novamber 1725. The second part, ff. 136-279, is a much older collection of Ṭūsī's recensions of the middle books. Autolycus' text, on ff. 241b–243a was copied in 672/1273. According to an ownership note on f. 136a, this second part

1. Elaheh Kheirandish, *The Arabic Version of Euclid's Optics*.
2. Regarding this MS, see Kheirandish, "A Report on Iran's 'Jewel' Codices of Tusi's Kutub al-Mutawasitat".
3. For this MS, see: Aydin Sayili, "Khwāja Naṣīr-i Ṭūsī wa raṣadkhāna-yi Marāgha," (in Persian) *Ankara Üniversitesi Dil ve Tarih-Coğrafya Fakültesi Dergisi* 14, no. 1-2 (1956): 11.

(apparently in 680/1281) was in possission of Barhebraus (d. 685/1286).

- **M (م):** Iran, Tabriz, Kitābkhāna-yi Millī-yi Tabrīz, 3484, pp. 58–63 (7th century).

Version V

- **Q (ق):** Iran, Mashhad, Kitābkhāna-yi Markazī Astān-i Quds, 5232.[1] This codex contains two of Maghribī's recensions of the middle books:
  Taḥrīr al-ukar li-Ṯāwdhūsiyūs, pp. 1–77;
  al-Kura al-mutaḥarrika, pp. 77–91.

The last folio of the codex, which contained the last page of Autolycus' text, is missing. The text ends at the middle of the proposition 14. This codex also bears an ownership note, on f. 1a, by Barhebraus, exactly like the one in MS H.

- **D (د):** Ireland, Dublin, Chester Beatty Library, Arabic, 3035.[2] This codex contains the following four items:
  Kitāb al-uṣūl li-Uqlīdis, Thābit ibn Qurra, 1a-126a;
  Taḥrīr al-ukar li-Ṯāwḏūsiyūs, Maghribī, 126b-144a;
  al-Kura al-mutaḥarrika, Maghribī, 144a-147b;
  Risāla fī al-uṣṭurlāb al-khaṭṭī, ff. 147b-149b.

The copyist's name and the copy date are mentioned at the end of the first text as Yusuf ibn Ibrāhīm ibn Abī al-Karam and Rabīʿ I 669/Novamber 1270. Maghribī's name appears at the beginning of *Taḥrīr al-ukar* as the author but not at the beginning of the Autolycus text.

- **R (ر):** Iran, Tehran, Kitābkhāna-yi Majlis-i Shurā-yi Islāmī, Shurā, 200. This was copied directly from MS D. We haven't collated this copy with our edition below. The Autolycus text is on ff. 254b–260a.

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1. This copy is introduced in Astān-i Quds' catalogue, volume 8, mistakenly under the shelfmark 5222. Autolycus' witness has a second reference number as 19293.
2 See: Gregg De Young, "Mathematical diagrams from manuscript to print: examples from the Arabic Euclidean transmission," *Synthese* 186, no. 1 (2012): 32.

Appendix 3.1: Editions of the Arabic Texts
Versions II, III, and V

V	III	II	
الشكل السابع كلّ دائرة عظيمة على بسيط كرة ثابتة عليها ومائلة على المحور تحدّ بين ظاهر الكرة وخفيها	ز **إذا كانت في كرة** دائرة عظيمة ثابتة عليها تحدّ بين[١] **ما يظهر منها وما لا يظهر**	ز **إذا كانت** دائرة **في كرة** تفصل بين **ما يظهر منها** وبين **ما لا يظهر**	
وكانت دوائر أخر متوازية قائمة على المحور وقاطعة للأفق	وكانت فيها دوائر قائمة **على المحور على زوايا قائمة تقطع الأفق**	وكان **على المحور** دوائر **على زوايا قائمة تقطع الأفق**	**1**
فطلوعها وغروبها على نقطٍ بأعيانها من[٢] الأفق	فإنّ طلوع تلك الدوائر **وغروبها** تكون أبداً[٣] **على نقطٍ بأعيانها من الأفق**	فطلوعها **وغروبها** دائماً **على نقطٍ بأعيانها من** النقط الّتي على **الأفق** كان الأفق مائلاً	
		أو كان على القطبين	
وميلها عليها ميلاً متساوياً.	فإن **كان الأفق مائلاً على المحور** كان ميلها على **الأفق** ميلاً متشابهاً.	وإذا **كان الأفق مائلاً عن المحور** فميل الدوائر المتوازية القائمة عن **الأفق** متساوية.	
مثاله ليكن دائرة ا ب ج د العظمى على بسيط كرة ثابتة عليها ومائلة على المحور، تحدّ بين ظاهر الكرة وخفيها	فليكن **في كرة دائرة** ا ب د ج [٤] تحدّ **بين**[٥] **ما يظهر منها وما لا يظهر**	فلنترك **في كرة دائرة** تفصل **بين ما يظهر منها** وبين **ما لا يظهر** وهي دائرة ا ب ج د [٦]	**2**
وليكن دائرتا ا ه ب ، ج ز د متوازيتان وقائمتان على المحور.	وليكن[٧] الدائرتان القائمتان **على المحور على زوايا قائمة** دائرتي ا ب ، ج د .	والدوائر القائمة **على المحور على زوايا قائمة** دائرتا **ا ب** ، **ج د** .	
فأقول إنّهما تطلعان[٨] أبداً على نقطتا ب ، ج وتغربان[٩] على	فأقول **إنّ** طلوع أجزاء **دائرتي ا ب** ، **ج د وغروبها** يكون أبداً **على نقطٍ بأعيانها من الأفق**، ويكون طلوعها	أقول **إنّ دائرتي ا ب** ، **ج د** [١١] طلوعهما وغروبهما دائماً **على نقطٍ بأعيانها من** النقط الّتي	**3**

	على **الأفق**، أمّا الطلوع فعلى **نقطتي ب ، د** وأمّا الغروب **فعلى نقطتي ا ، ج** .	إذا مرّت **بنقطتي د ، ب** [١٠] وغروبها إذا مرّت ب**نقطتي ا ، ج** .	نقطتا ا ، د .
			وإنّ ميلهما على الأفق ميلاً متساوياً.
4	**فإن لم يكن** ذلك **كذلك فليكن** أن **أمكن طلوع دائرة ا ب** على نقطة أخرى وهي ه	**فإن لم يكن** الأمر كذلك **وأمكن** غيره فليكن **طلوع** بعض **دائرة ا ب** إذا مرّت بنقطة ه	برهانه إن لم تطلع[١٢] دائرة ا ب على ب فلتطلع على م .
	وغروبها على **نقطة ا** .	**وغروبها** إذا مرّت ب**نقطة ا** .	
	وليكن قطب **الدوائر المتوازية نقطة ز** ،	وليكن[١٣] أحد قطبي **الدوائر المتوازية نقطة ز** ،	وليكن قطب الكرة ل
	ونرسم على نقطة ز و**قطبي دائرة ا ب د ج** دائرة عظيمة **وهي دائرة ح ز ط** .	ولنرسم دائرة عظيمة تمرّ بنقطة ز وب**قطبي**[١٤] **دائرة** ا ب د ج [١٥] **وهي دائرة ح ز ط** .	ونرسم دائرة عظيمة تمرّ بالقطب وبقطب دائرة ا ب ج د وهي دائرة ى ل ك .
			فهي قائمة على الأفق وعلى كلّ المتوازيات.
	ونصل **خطوط** ح ز ، ح ط ، ز ه .	ولنوصل **خطوط** ح ط ، ح ز ، ز ه ، ز ب المستقيمة.	وأيضاً نصل خطوط ل ى ، ل م ، ل ب .
5			فخطّ ل م كخطّ ل ب من اجل أنّ دائرة ا ب تطلع على م وخروجهما من القطب إلى المحيط.
	فلأنّ في الكرة دائرة عظيمة، هي **دائرة ح ز ط** ، قد قطعت من الدوائر الّتي فيها، وهي **دائرة ا ب د ج** ،[١٨] على قطبيها، فقطعها لها **بنصفين على زوايا قائمة. فخطّ ح ط قطر دائرة ا ب د ج** ،[١٩]	**فلأنّ دائرة ح ز ط** العظمى في كرة وهي تقطع **دائرة** ا ب د ج [١٦] من الدوائر الّتي في الكرة وتمرّ بقطبيها وهي تقطعها **بنصفين وعلى زوايا قائمة. فخطّ ح ط قطر دائرة** ا ب د ج ،[١٧]	

	ودائرة ح ز ط قائمة على دائرة ا ب د ج ،[22] **على زوايا قائمة.**	**ودائرة ح ز ط** [20] **قائمة على دائرة ا ب د ج ،**[21] **على زوايا قائمة.**	وأيضاً فلأنّ دائرة ی ل ك قائمة على دائرة ا ب ج د
	فلأنّه قد قام **على قطر دائرة** ا ب د ج ،[25] وهو خطّ ح ط ، **قطعة ح ز ط على زوايا قائمة،**	فقد تحمّل[23] **على قطر** ح ط من **دائرة** ا ب د ج [24] **قطعة ح ز ط** وهي قائمة عليها **على زوايا قائمة،**	
	وانقسمت قوس القطعة الّتي قامت وهي **قوس ح ز ط** ، **بقسمين** مختلفين **على نقطة ز** ،	وقد قسمت **قوس ح ز ط بقسمين** غير متساويين **على نقطة ز** ،	
	وقوس[27] **ز ح أصغر من** النصف،	**وقوس ز ح أصغر من** نصفها،	وقوس ی ل أقلّ من[26] نصف قوس ی ك ،
	فخطّ ح ز إذن أصغر من **جميع الخطوط** تقع **من نقطة** ز على دائرة **ا ب د ج** .[31] والخطّ **الأقرب** من **خطّ ز ح** ، أصغر **من الخطّ الأبعد منه.**	**فخطّ**[29] **ز ح** أقصر **جميع الخطوط** الّتي تخرج **من نقطة ز** وتلقى الخطّ المحيط بدائرة ا ب د ج .[30] فالخطّ **الأقرب** إلى **خطّ ز ح** هو أقصر **من الخطّ الأبعد منه.**	فخطّ ی ل أقصر الخطوط الخارجة من نقط ل إلى محيط دائرة ا ب ج د وما قرب منه أصغر[28] ممّا بعد.
	فخطّ ز ه إذن أصغر **من خطّ ز ب ولكنّه مساوٍ له،** هخ [= هذا خلف] لا يمكن.	**فخطّ ز ه** أقصر **من خطّ ز ب ولكنّه مساوٍ له،** وذلك محال.	فخطّ ل م أقصر من خطّ ل ب وقد كانا[32] متساويين هذا محال.
	فليس يطلع **دائرة ا ب** على **نقطة أخرى** سوى **نقطة ب** .	**فليس** يكون طلوع شيء من **دائرة ا ب** عند ممرّها ب**نقطة أخرى** غير **نقطة ب** .	فلا يطلع دائرة ا ب على غير نقطة ب .
6	**ولا** تغرب على نقطة **أخرى** خلا **نقطة ا** .	**ولا** غروبها أيضاً يكون عند ممرّها بنقطة **أخرى** غير **نقطة ا** .	
	وكذلك أيضاً نبيّن دائرة ج د تطلع على **نقطة د**	**وكذلك أيضاً نبيّن** أنّ طلوع جميع أجزاء **دائرة ج د** يكون عند ممرّها ب**نقطة د**	وهذا القول على دائرة ج د أنّها لا تطلع على غير نقطة ج .
	وتغرب على نقطة ج .	وغروبها يكون عند ممرها بنقطة ج .	

	فيجب من ذلك أن يكون دائرتا ا ب ، ج د تطلعان وتغربان على نقطٍ **بأعيانها من** النقط الّتي على **الأفق**.	فطلوع جميع أجزاء دائرتي **ا ب** ، **ج د**[٣٥] وغروبها يكون أبداً على نقط **بأعيانها من** دائرة **الأفق**.	فهما تطلعان[٣٣] أبداً على نقطتين بعينهما من الأفق وتغربان[٣٤] على نقطتين بعينهما.
7	**ومن البيّن** أيضاً **إن** جعلنا **القطب على دائرة ا ب د ج**[٣٦]،	**ومن البيّن** أمّا **إن** وضعنا **القطب على دائرة ا ج د ب** ،	
		وصيّرناه نقطة ح ،	
	إنّ **دائرة ا ب** على ذلك المثال تطلع على نقطة ب وتغرب على نقطة ا ،	فإنّ طلوع **دائرة ا ب** يكون عند ممرّها بنقطة ب[٣٧] وغروبها عند ممرّها بنقطه ا ،[٣٨]	
	لأنّها متى طلعت على **نقطة ه**	وذلك أنّها إن طلعت عند ممرّها ب**نقطة ه**	
	صار خطّ ه ح أيضاً أصغر من خطّ ب ح ، لكن خطّ ه ح مساوٍ[٣٩] ل ح ب وذلك لأنّهما **من القطب**	يصير أيضاً خطّ ح ه مساوياً لخطّ ح ب ، وذلك أنّهما خرجا **من القطب** إلى الخطّ المحيط بالدائرة	
	هخ [= هذا خلف] لا يمكن.	وذلك محال.	
8	وأيضاً فليكن **دائرة** ا ب د ج **مائلة عن المحور**.	فليكن[٤٠] **دائرة** ا ب د ج[٤١] **مائلة على المحور**.	
	أقول **إنّ ميل دائرتي ا ب** ، د ج عن[٤٣] الأفق متساوية.	فأقول **إنّ ميل دائرتي ا ب** ، ج د على أفق ا ب د ج[٤٢] ميل متشابهة.	وأقول أيضاً إنّ ميلهما على الأفق ميلٌ متساوٍ.
	برهانه إنّا نصلت **خطوط ا ب** ، **ج د** ، ك خ ،[٤٦] ل م .	فلنوصل[٤٤] **خطوط ا ب** ، **ج د** ، ك م ، ل ن[٤٥] المستقيمة.	برهانه إنّا نخرج الفصول المشتركة للدوائر كلّها وهي خطوط ا ب ، ج د ، ى ك ، ه ح ، ز ط .
9	**فلأنّ دائرة ح ز ط تقطع** دوائر **ا ج د ب** ، ا ب ، ج د على أقطابها، فقطعها لها بنصفين **وعلى زوايا قائمة**،	**فلأنّ دائرة ح ز ط تقطع** دائرتي **ا ج د ب** ، ا ك ب ، وتمرّ بقطبيها وهي[٤٧] تقطعها **على زوايا قائمة**،	

			فلأنّ دائرة ا ه ب موازية لدائرة ج ز د وفصلهما سطحا دائرتي ى ه ز ك ، ا ب ج د فخطّ ه ح موازٍ لخطّ ز ط وزاوية ه ح ط كزاوية ز ط ك ،
	فدائرة ح ز ط قائمة على كلّ واحد من دوائر **ا ج د ب** ، ج د ، ا ب **على زوايا قائمة.** فيجب إذن تكون كلّ **واحدة من** دوائر **ا ج د ب** ، ا ب ، ج د **قائمة على دائرة ح ز ط** [50] **على زوايا قائمة.**	**فدائرة ح ز ط قائمة على كلّ** واحدة **من** دائرتي **ا ج د ب** ، ا ك ب **على زوايا قائمة**. فكلّ **واحدة**[48] **من** دائرتي **ا ج د ب** ، ا ك ب أيضاً **قائمة على دائرة** ح ز ط [49] **على زوايا قائمة.**	وأيضاً فلأنّ دائرة ى ه ك قائمة على الدوائر كلّها، فهي أيضاً قائمة عليها،
	فالفصلان المشتركان بينهما وهما خطّا ا ب ، ج د قائمان **على دائرة ح ز ط على زوايا قائمة.**	فالفصل المشترك لهما الّذي هو خطّ ا ب قائم **على دائرة ح ز ط على زوايا قائمة.**	فخطّا ب ح ، ج ط قائمان[51] على سطح دائرة ى ه ز .
	فهما أيضاً قائمان **على جميع الخطوط الّتي في سطح ح ز ط** الّتي تماسّهما، **على زوايا قائمة.** وتماسّ خطّ ا ب كلّ واحد من ح ط ،[54] ل م وهما في سطح دائرة ح ز ط . **فخطّ ا ب قائم على كلّ واحد من خطّي ح ط** ، ل م **على زوايا قائمة.**	فهو قائم **على جميع الخطوط الّتي** تخرج منه ويكون **في سطح ح ز ط** ، **على زوايا قائمة.** وكلّ واحد من خطّي ح ط ، ك م اللّذين هما في سطح دائرة ح ز ط يخرج[52] من موضع من خطّ[53] ا ب . **فخطّ ا ب قائم على كلّ واحد من خطّي ح ط** ، ك م **على زوايا قائمة.**	
			فكلّ واحدة من زوايا ب ح ه ، ب ح ط ، ج ط ز ، ج ط ك قائمة.

فزاوية ه ح ط ميل[55] دائرة ا ه ب على الأفق أعني دائرة ا ب ج د .	فزاوية ك م ط هي الميل الّذي تميل به **دائرة ا ب** على **دائرة** ا ب د ج .[56]	فتصير زاوية **ل م ط** [57] ميل **دائرة ا ب** عن **دائرة** ا ب د ج .	
وزاوية ز ط ك ميل[58] دائرة ج ز د على الأفق.	ولهذه الأشياء بأعيانها **تكون زاوية** ل ن ط [59] الميل الّذي تميل به **دائرة ج د** على دائرة ا ب د ج .[60]	وكذلك **تكون زاوية** خ ك ط [61] ميل **دائرة ج د** عن دائرة ا ب د ج .	
	ولأنّ سطحي **ا ب** ، **ج د** المتوازيين قد قطعا بسطح[62] وهو سطح ح ز ط ، يكونُ الفصلان المشتركان لهما اللّذان هما خطّا ك م ، ل ن ،[63] متوازيين، فزاوية ك م ط **مساوية لزاوية** ل ن ط .[64]	**ولأنّ** بسيطي[65] سطحين متوازيين وهما **ا ب** ، **ج د** ، تقطعهما بسيط سطح وهو دائرة ح ز ط ، فالفصول المشتركة بينهما وهي ك خ ،[66] ل م ، متوازية، فتصير زاوية ل م ط **مساوية لزاوية** خ ك ط .[67]	
	ويقال أنّ السطح يميل على سطح ميلاً شبيهاً بميل سطح آخر على سطح آخر[68] إذا كانت الخطوط الّتي تخرج إلى الفصول المشتركة لها[69] على زوايا قائمة تحيط في كلّ واحد من السطحين بزوايا متساوية. وزاوية ك م ن مساوية لزاوية ل ن ط .		**10**
	وزاوية ك م ط هي الميل الّذي يميل به **دائرة ا ب** على **دائرة** ا ب د ج .[70] **وزاوية** ل ب ز ط هي الميل الّذي يميل به **دائرة ج د** على **دائرة ا ب د ج** .[71]	**وزاوية** ل م ط فميل **دائرة ا ب** عن **دائرة** ا د ج **وزاوية** خ ك ط ميل **دائرة ج د** عن **دائرة ا ب د ج** .	
لكن[72] الزاوية كالزاوية فالميل كالميل وهكذى القول على كلّ الدوائر المتوازية وهو المطلوب.	فميل دائرتي **ا ب** ، **ج د** على **دائرة ا ب د ج** [73] متساوية **وذلك ما أردنا أن نبيّن.**	فدائرتا **ا ب** ، **ج د** إذن مائلتان عن **دائرة ا ب د ج** ميلاً متشابهاً **وذلك ما أردنا أن نبيّن.**	

Appendix 3.2: Editions of the Arabic Texts
Versions I and IV

IV	I	
ز إذا كانت دائرة الأفق **مائلة على المحور**	**ز إذا كانت** على كرة **دائرة** عظيمة ثابتة عليها تحدّ بين ظاهر الكرة وخفيها وهي **مائلة على المحور**	1
وقطعتها **دوائر** يكون المحور عموداً عليها	وكانت **دوائر** أخرى[٧٤] تقطع الأفق وهي قائمة على المحور[٧٥]	
كان طلوع النقط الّتي تكون على تلك الدوائر وخفائها على الأفق **على** نقطٍ **بأعيانها**	فإنّ طلوعها وغروبها يكونان[٧٦] من الأفق **على** نقطة واحدة **بأعيانها**	
وميل تلك الدوائر على الأفق **ميلاً متشابهاً**.	وميلها من[٧٧] الأفق يكون[٧٨] **ميلاً متشابهاً**.	
فليكن الأفق ا ب ج د وهي **مائلة على المحور**	مثال ذلك أن[٧٩] نتوهّم على كرة دائرة عظيمة، ثابتة عليها، تحدّ بين ظاهر الكرة[٨٠] وخفيها وهي دائرة ا ب ج د ، **مائلة على المحور**.	2
ودائرتا ب ه د ، **ز ح ط** قاطعتين[٨١] للأفق والمحور عمود عليهما.	وليكن[٨٢] **دائرتا ب ه د** ، **ز ح ط** [٨٣] قائمة على المحور[٨٤] تقطع الأفق.	
	فأقول إنّ دائرتي ب ه ، ز ح [٨٥] طلوعها وغروبها من الأفق على نقطة[٨٦] واحدة بأعيانها ويكون ميلهما[٨٧] عليه ميلاً[٨٨] متشابهاً.	3
وليكن الأفق مماسة ل**دائرتي ا ك ل** ، **ج م** .	برهان ذلك من اجل أنّ دائرة ا ب ج د مائلة على المحور فإنّها تماسّ دائرتين متساويتين متوازيتين قطباهما قطب[٨٩] الكرة فليكن[٩٠] هاتان الدائرتان[٩١] **دائرتي ا ك ل** ، **ج م** .[٩٢]	4
وليكن القطب **الظاهر س** .	**وليكن** قطب الكرة **الظاهر**[٩٣] نقطة **س**	
	وهو أيضاً قطب دائرة ا ك ل .[٩٤]	
وترسم **على ا** ، **س دائرة عظيمة فهي** تمرّ بقطب دائرة[٩٥] **ا ب ج د** ، **وتكون قائمة عليها** على قوائم.	ولنرسم[٩٦] **على** نقطتي ا ، **س دائرة عظيمة فهي**[٩٧] **لا** محالة تجوز على قطبي[٩٨] **ا ب ج د** ، **وتكون**[٩٩] **قائمة عليها**.	
ولكونها مارّة بقطب **دائرة ج م** تمرّ ب**نقطة ج** .	ومن اجل أنّها[١٠٠] تجوز على قطبي **دائرة**[١٠١] **ج م فهي** إذن[١٠٢] تجوز على **نقطة ج** .	
ولتكن هي **دائرة ا س ك ه ح ج م** .[١٠٣]	ف**لتكن** ك**دائرة ا س ك ه ح ج** .[١٠٤]	

ولتكن الفصول **المشتركة للسطوح**[١٠٥] **ب ف د** ، ز ع ط ، ا ج ، ك ا ، ف ه ، ع ح .[١٠٦]	ولنخرج التقاطعات[١٠٧] **المشتركة للسطوح** وهي خطوط **ب ف د** ، ا ف ع ج ، ز ع ط ، ه ف ، ع ح ، ا ك .[١٠٨]	**5**
	ومن اجل أنّ دائرة ا ك ل [١٠٩] قطباها[١١٠] قطبا[١١١] الكرة فإنّها قائمة على المحور. ودائرتا[١١٢] ب ه د ، ز ح ط [١١٣] قائمتان[١١٤] على المحور،[١١٥]	**6**
ولتوازي دوائر ا ك ، ب د ، ز ط	فدوائر[١١٦] ا ك ،[١١٧] ز ح ط ، ب ه د [١١٨] متوازية	
	ويقطعها[١١٩] سطح واحد وهو سطح دائرة ا س ك ه ح ج ؛[١٢٠] فإنّ[١٢١] التقاطعات[١٢٢] المشتركة لها متوازية،	
تكون فصول **ا ك** ، **ف ه** ، **ع ح متوازية**.	فإذن[١٢٣] خطوط **ا ك** ، **ف ه** ، **ع ح** [١٢٤] **متوازية**.	
فـزاوية ف ا ك مساوية لزاوية ع ف ه . **وزاوية ف ا ك حادّة**، فـ**زاوية ع ف ه حادّة**.	فإذن[١٢٥] **زاوية ف ا ك مساوية لزاوية ع ف ه وزاوية ف ا ك حادّة** فإذن[١٢٦] **زاوية**[١٢٧] **ع ف ه حادّة**.	
ونقول **إنّ دائرة ب ه د لا تلقى** في دورتها من **دائرة ا ب ج د غير نقطتي ب** ، **د** .	فأقول **إنّ دائرة ب ه د** إذا دارت الكرة لا تمكن أن **تلقى دائرة ا ب ج د** على **غير نقطتي ب** ، **د** .	**7**
وإلّا فلتلقها[١٢٨] على **ق**	فإن أمكن فتلقاها[١٢٩] نقطة أخرى كنقطة **ق**	**8**
ونصل **س ق** ، **س د**	ونخرج من قطب س خطّي **س ق** ، **س د** [١٣٠]	
فيكونان **متساويين**.[١٣١]	فخطّا[١٣٢] س د ، ق س [١٣٣] إذن[١٣٤] **متساويين**.[١٣٥]	
ولأنّ قطعة ا ه ج ، على قطر ا ج ، قائمة على دائرة ا ب ج د و ا س أصغر من نصفها، يكون وتر ا س أقصر خطّ يخرج من س إلى محيط دائرة ا ب ج د .	ومن اجل أنّ نقطة س هي القطب، فإنّ قوس س ا [١٣٦] مساوية لقوس س ك .[١٣٧] فإذن[١٣٨] قوس ج ح ه ك س [١٣٩] أعظم من قوس س ا ؛[١٤٠] ومن اجل أنّ دائرة ا ب ج د قد قامت عليها قطعة من دائرة وهي قطعة ج ا ،[١٤١] على قطرها وفصل منها قوس س ا [١٤٢] وهي أقلّ من نصفها، فإذن[١٤٣] الخطّ الّذي يوترها هو أقصر الخطوط الّتي تخرج من تلك النقطة وتلقى[١٤٤] دائرة ا ب ج د وما قرب منه[١٤٥] كان أقصر ممّا بعد منه.[١٤٦]	

و **س ق أقصر من س د** وكانا متساويين، **هذا خلف.**	فإذن[147] خطّ **س ق أقصر من** خطّ **س د** ، وقد كان مساوياً[148] له **هذا خلف لا** يمكن.	**9**
فإذن طلوع النقط الّتي على **دائرة ب ه د** وغروبها **لا** يكون على **غير نقطتي**[149] **ب** ، **د** .	**فإذن**[150] **دائرة**[151] **ب ه د لا** تلقى دائرة ا ب ج د على **غير نقطتي ب** ، **د** .	
	فإذن[152] طلوعها وغروبها من الأفق يكون أبداً على هاتين النقطتين. وكذلك أيضاً دائرة ز ح ط .[153]	
	فأقول[154] إنّ دائرتي[155] ب ه د ، ز ح[156] يكون ميلها على دائرة ا ب ج د ميلاً متشابهاً.	**10**
وأيضاً لأنّ دائرة ا ه ج تمرّ بقطبي **دائرتي ا ب ج د** ، **ب ه د** المتقاطعتين فهي تنصف قطعهما،	برهان ذلك من اجل أنّ **دائرتي ا ب ج د** ، **ب ه د**[157] تتقاطعان وقد رسم على أقطابهما[158] دائرة عظيمة وهي دائرة ا س ك ه ،[159] فهي[160] إذن[161] تقطع قسيهما[162] على أنصافها[163]	**11**
ف ا ب ، ا د متساويان[164]	فإذن[165] قوس د ا متساوية[166] لقوس ا ب	
وكذلك ب ه ، ه د .	وقوس ب ه[167] متساوية[168] لقوس[169] د ه .[170]	
	ومن اجل أنّ قوس ا ب مساوية لقوس ا د وقد أخرج قطر ا ف ع ج ،[171]	
وقطر ا ج ينصف ب د على ف ويكون عموداً عليه. ولتساوي قوسي ب ه ، ه د		
وخطّي[172] **ب ف** ، **ف د**	فإنّ خطّ **ب ف** مساوٍ لخطّ **ف د**	
يكون ه ف أيضاً عموداً على ب د .	وزاوية د ف ه قائمة	
	وأيضاً من اجل أنّ خطّ ب ف مساوٍ لخطّ ف د[173]	
	وقوس ب ه مساوية لقوس ه د	
	وقد أخرج خطّ ف ه	
	فإنّ[174] زاوية د ف ه قائمة.	

	ومن اجل أنّ سطحي ا ب ج د ،[١٧٥] ب ه د يتقاطعان وقد قام على[١٧٦] تقاطعهما المشترك لهما، أمّا في سطح ا ب ج د [١٧٧] فخطّ ف ع ج [١٧٨] وأمّا في سطح ب ه د فخطّ ف ه ، وهذان[١٧٩] الخطّان يحيطان بزاوية حادّة وهي زاوية ع ف ه ،	
ولكون ف ه ، ف ج عمودين على فصل ب د وهما في سطحي دائرتي ا ب ج د ، ب ه د		
تكون **زاوية** ه ف ج **هي ميل سطح** دائرة **ب ه د على سطح** دائرة **ا ب ج د** .	فإذن[١٨٠] **زاوية** ع ف ه [١٨١] **هي**[١٨٢] **ميل سطح ب ه د على سطح ا ب ج د** .	
وكذلك زاوية ج ع ح هي ميل سطح دائرة **ز ح ط على سطح دائرة ا ب ج د** .	**وكذلك** تكون **زاوية ج ع ح** [١٨٣] **هي**[١٨٤] **ميل سطح ز ح ط** [١٨٥] **على سطح ا ب ج د** .[١٨٦]	
ولتساوي[١٨٧] زاويتي ه ف ج ،[١٨٨] ح ع ج	وزاوية ع ف ه مساوية لزاوية[١٨٩] ج ع ح .[١٩٠]	
يكون الميلان متشابهين وذلك ما أردناه.	فإذن[١٩١] ميل سطحي[١٩٢] ب ه ، ح ز [١٩٣] ميلاً متشابهاً،[١٩٤] وذلك ما أردنا أن نبيّن.[١٩٥]	

Appendix 3.3: Apparatus

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١. دائرة عظيمة ثابتة عليها تحدّ بين] ل = دوائر تحدّ: س.
٢. من] + من: ق.
٣. تكون أبداً] ل = ابداً تكون: س.
٤. ا ب د ج] ا ج ب د ح : س = اح بد : ل.
٥. بين] –س.
٦. ا ب د ج] ابجد : ف.
٧. وليكن] ولتكن: س.
٨. تطلعان] ىطلعان: د = يطلعان: ق.
٩. تغربان] ىغربان: د = يغربان: ق.
١٠. د ، ب] ب د : ل.
١١. ا ب ، ج د] ابجد ج د : ف.
١٢. تطلع] د = يطلع: ق.
١٣. وليكن] ل = ولكن: س.
١٤. وبقطبي] وىقطتى: س.
١٥. ا ب د ج] ابج د : س = اب ج د : ل.
١٦. ا ب د ج] ابج د : س = اب حد : ل.
١٧. ا ب د ج] ابجد : س = ا ب ج د : ل.
١٨. ا ب د ج] ا ب ر ج : ف.
١٩. ا ب د ج] ا ب ر ج : ف.
٢٠. ح ز ط] ج ز ط : س = حر ط : ل.
٢١. ا ب د ج] ابج د : س = ابجد : س.
٢٢. ا ب د ج] ا ب ر ج : ف.
٢٣. تحمّل] س = عمل: ل.
٢٤. ا ب د ج] ابج د : س = ابجد : ل.
٢٥. ا ب د ج] ا ب ر ج : ف.
٢٦. من] + من: د.
٢٧. وقوس] د قوس: ف.
٢٨. أصغر] د = اقصر (متغيّر من «اصغر»): ق.
٢٩. فخطّ] + ا : س.
٣٠. ا ب د ج] ابج د : س = ابجد : ل.
٣١. ا ب د ج] ا ب ر ج : ف.
٣٢. كانا] د = كان: ق.
٣٣. تطلعان] يطلعان: د، ق.
٣٤. وتغربان] ويغربان: د، ق.
٣٥. وغروبها يكون عند ممرها بنقطة ج فطلوع جميع أجزاء دائرتي ا ب ، ج د] –س.
٣٦. ا ب د ج] ا ب ر ج : ف.

٣٧. ب] ل = ا : س.
٣٨. وغروبها عند ممرّها بنقطة آ] ل = –س.
٣٩. مساوٍ] + (شيء غير مقروء) : ف.
٤٠. فليكن] س = وليكن: ل.
٤١. ا ب د ج] ابج د : س = ابجد : ل.
٤٢. ا ب د ج] ابج د : س = ابجد : ل.
٤٣. عن] ف (متغيرّ من «على»).
٤٤. فلنوصل] س = فليصل: ل.
٤٥. ل ن] كد : س = ل ز : ل.
٤٦. ك خ] ك ح : ف.
٤٧. وهي] س = فهي: ل.
٤٨. واحدة] س = واحد: ل.
٤٩. ح ز ط] جرط : س.
٥٠. ح ز ط] ه ب ط : س.
٥١. قائمان] قائمتان: د، ق.
٥٢. يخرج] –ل.
٥٣. خطّ] ل = –س.
٥٤. ح ط] ف (متغيرّ من شيء غير مقروء).
٥٥. ميل] مثل: د.
٥٦. ا ب د ج] ابج د : س = ابجد : ل.
٥٧. ك م ط] ل م : ف.
٥٨. ميل] مثل: د.
٥٩. ل ن ط] لب ر ط : س = لر ط : ل.
٦٠. دائرة ا ب د ج] دائرة ابجد : ل = اب ح د : س.
٦١. خ ك ط] ح ك ط : ف.
٦٢. بسطح] + ما: ل.
٦٣. ل ن] ل ر : س.
٦٤. ل ن ط] لرط : س.
٦٥. بسيطي] ف (متغير من «بسيطين»).
٦٦. ك خ] ك ح : ف.
٦٧. خ ك ط] ح ك ط : ف.
٦٨. على سطح آخر] ل = –س.
٦٩. لها] –ل.
٧٠. ا ب د ج] ا ب ح د : س = ابجد : ل.
٧١. ا ب د ج] اب ج د : س.
وزاوية ل ب ز ط هي الميل الّذي يميل به دائرة ج د على دائرة ا ب د ج] –ل.
٧٢. لكن] د = لالن: ق.
٧٣. ا ب د ج] اب ج د : س = ابجد ميل: ل.

۷۴. أخرى] اخر: خ، ك.
۷۵. المحور] + وكانت دو: آ (مشطوب).
۷۶. يكونان] يكون: خ، ك.
۷۷. من] على: خ.
۷۸. يكون] -ك.
۷۹. أن] انا: ك.
۸۰. ظاهر الكرة] ظاهرها: خ، ك.
۸۱. قاطعتين] قاطعن: ج.
۸۲. وليكن] + دوائر قائم وهي: آ (مشطوب) = دواير اخر وهي: خ.
۸۳. ز ح ط] طزح نكون: خ.
۸۴. وليكن دائرتا ب ه د ، ز ح ط قائمة على المحور] -ك.
۸۵. ب ه ، ز ح] بهد طزح نكون: خ = بهد زح ط يكون: ك.
۸۶. نقطة] نقطٍ: ك.
۸۷. ميلهما] ميلها: آ، خ، ك.
۸۸. ميلاً] -خ.
۸۹. قطباهما قطب] قطباها قطبا: خ = اقطابهما قطبا: ك.
۹۰. فليكن] فلكونا: خ = فلتكن: ك.
۹۱. هاتان الدائرتان] -خ.
۹۲. ا ك ل ، ج م] ا ل ك ج م : خ.
۹۳. قطب الكرة الظاهر] ك = + هو: خ = القطب الكرة ظاهر: آ.
۹۴. ا ك ل] ا ل ك : خ.
۹۵. دائرة] (تحت السطر في «ح»).
۹۶. ولنرسم] ولترسم: خ.
۹۷. فهي] وهى: خ، ك.
۹۸. قطبي] + دايرة: خ، ك.
۹۹. وتكون] فكون: خ.
۱۰۰. ومن اجل أنّها] ومن اجل ذلك ابصا: خ.
۱۰۱. دائرة] + ابجد : ك.
۱۰۲. إذن] اذا: خ، ك.
۱۰۳. ا س ك ه ح ج م] ا س ك ه ج م : ح.
۱۰۴. ا س ك ه ح ج] ا س ك ه ز ج : خ.
۱۰۵. للسطوح] السطوح: م.
۱۰۶. ب ف د ، ز ع ط ، ا ج ، ك ا ، ف ه ، ع ح] ب ف د ر ع ط ا ج ا ك ع ح ه ف : م.
۱۰۷. التقاطعات] التقاطعان: آ، ك (متغير من «التقاطعات») = المقطعات: خ.
۱۰۸. ب ف د ، ا ف ع ج ، ز ع ط ، ه ف ، ع ح ، ا ك] ب ف ع ط عز فه ا ك : خ = فه ع ح ا ك ا ج بد رط : ك.
۱۰۹. ا ك ل] ا ل ك : خ.
۱۱۰. قطباها] قطباهما: آ = قطباهاهما: ك.
۱۱۱. قطبا] قطب: آ.

١١٢. ودائرتا] فدايرتي: ك.
١١٣. زح ط] ح ز ط : خ.
١١٤. قائمتان] قايمتين: ك.
١١٥. المحور] + ايصا: خ.
١١٦. فدوائر] ودايرة: ك.
١١٧. ا ك] ا لك : خ = ا ك ل : ك.
١١٨. زح ط ، ب ه د] بهد ح ز ط : خ.
١١٩. ويقطعها] وتقطعهما: خ.
١٢٠. ا س ك ه ح ج] اسك هزج : خ.
١٢١. فإنّ] فاذا: خ، ك.
١٢٢. التقاطعات] المقطعات: خ.
١٢٣. فإذن] فاذا: خ، ك.
١٢٤. ع ح] عز : خ.
١٢٥. فإذن] فاذا: خ، ك.
١٢٦. فإذن] فاذا: خ.
١٢٧. فإذن زاوية] فزاوية: ك.
١٢٨. فلتلقها] م = فَلْتَلْقَها: ح = فلىقطعها: ج.
١٢٩. فتلقاها] فليلقها على: خ = + على: ك.
١٣٠. س د] ش د : آ.
١٣١. متساويين] متساوسن: ج.
١٣٢. فخطّا] فخطي: ك.
١٣٣. س د ، ق س] ش د ق س د ا : آ = سق سد: خ، ك.
١٣٤. إذن] اذا: خ، ك.
١٣٥. متساويين] متساويان: خ، ك.
١٣٦. س ا] ش ا : آ.
١٣٧. س ك] ش ك : آ.
١٣٨. فإذن] فاذا: خ، ك.
١٣٩. ج ح ه ك س] ج ك ه ك س : آ = ج ز ك : خ = ح ج ه ل س: ك.
١٤٠. س ا] ش ا : آ.
١٤١. ج ا] ج ه ا : ك.
١٤٢. س ا] ش ا : آ = -ك.
١٤٣. فإذن] فان: خ = فاذا: ك.
١٤٤. وتلقى] فيلقى: ك.
١٤٥. منه] منها: ك.
١٤٦. منه] عنه: خ.
١٤٧. فإذن] فاذا: خ، ك.
١٤٨. مساوياً] مساوٍ: ك.
١٤٩. نقطتي] نقط: م، ح.

١٥٠. فإذن] فاذ: خ = فاذا: ك.
١٥١. دائرة] دائر: آ.
١٥٢. فإذن] فاذا: خ، ك.
١٥٣. ز ح ط] ح ز ط: خ.
١٥٤. فأقول] واقول: ك.
١٥٥. دائرتي] دواير: خ.
١٥٦. ز ح] ح ز ط : خ = ز ح ط: ك.
١٥٧. ب ه د] ب ه ح : آ = بهز: ك.
١٥٨. أقطابهما] اقطابها: ك.
١٥٩. ا س ك ه] ا ش ك ه : آ = اسهج : خ = سهج : ك.
١٦٠. فهي] هي: ك.
١٦١. إذن] اذا: خ، ك.
١٦٢. قسيهما] قسيها: ك.
١٦٣. أنصافها] انصافهما: خ.
١٦٤. متساويان] متساويين: ح.
١٦٥. فإذن] فاذا: خ، ك.
١٦٦. متساوية] مساوية: خ، ك.
١٦٧. ب ه] د ه: خ.
١٦٨. متساوية] -خ = مساوية: ك.
١٦٩. لقوس] -ك.
١٧٠. د ه] ه ب : خ = هد: ك.
١٧١. ا ف ع ج] افه عر : خ = ا ع فج : ك.
١٧٢. وخطّي] ووتر ك: ح.
١٧٣. وزاوية د ف ه قائمة وأيضاً من اجل أنّ خطّ ب ف مساوٍ لخطّ ف د] -خ.
١٧٤. فإنّ] فاذا: خ.
١٧٥. ا ب ج د] ابد : خ.
١٧٦. على] + نقطة: ك.
١٧٧. ا ب ج د] ابج : خ.
١٧٨. ف ع ج] عف : خ.
١٧٩. وهذان] فهذان: خ = وهاذان: ك.
١٨٠. فإذن] فان: خ.
١٨١. فإذن زاوية ع ف ه] -ك.
١٨٢. هي] -خ.
١٨٣. ج ع ح] جعز : خ.
١٨٤. هي] -خ.
١٨٥. ز ح ط] ح ط : خ.
١٨٦. وكذلك تكون زاوية ج ع ح هي ميل سطح ز ح ط على سطح ا ب ج د] -آ.
١٨٧. ولتساوي] + ولتساوي: م.

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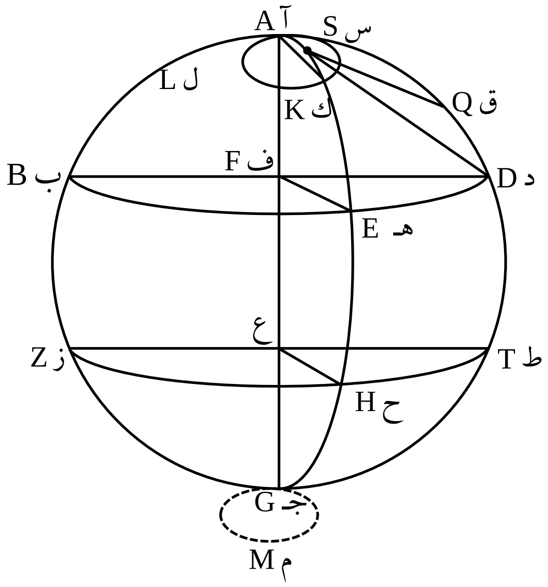
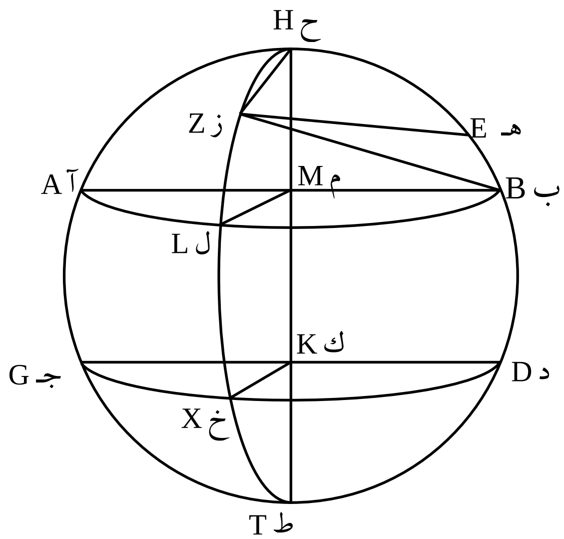
[١٨٨]. ه ف ج] ه ب ج: ح.
[١٨٩]. مساوية لزاوية] مثل زاوية: خ.
[١٩٠]. ج ع ح] جعز : خ.
[١٩١]. فإذن] فاذا: خ، ك.
[١٩٢]. سطحي] سطح: آ.
[١٩٣]. ب ه ، ح ز] بهد ح ز ط على سطح ابجد : خ = ب ه ، ز ح ط على سطح دائرة ابجد : ك.
[١٩٤]. ميلاً متشابهاً] مىل مساو: خ.
[١٩٥]. نبيّن] بيانه: ك.

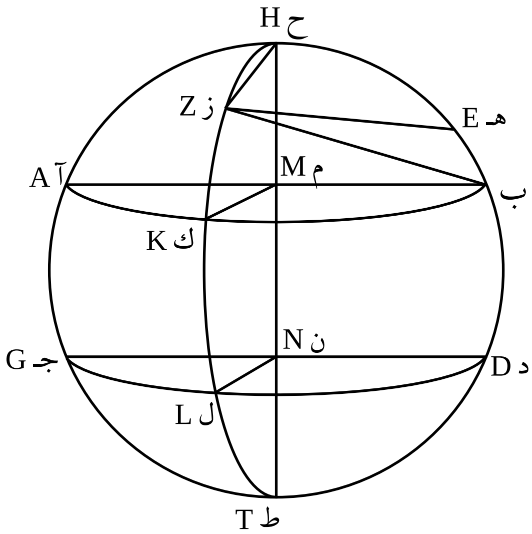
Appendix 3.4: Figures



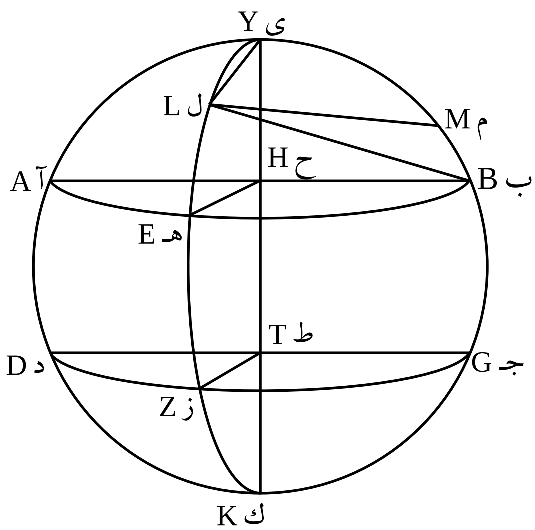
Versions I & IV



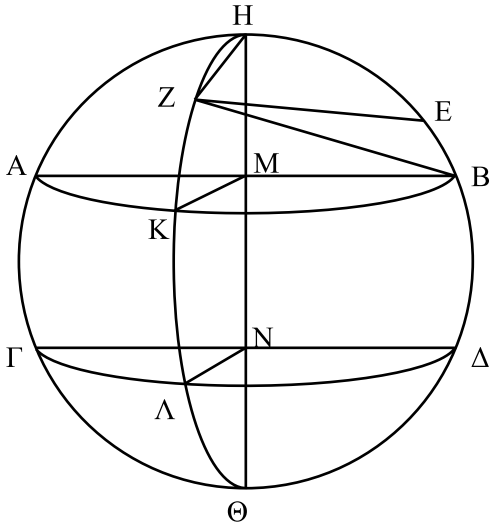
Version II



Version III



Version V



Greek Version

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